35th Annual West River Mathematics Contest

April 13, 1985

West River Math Contest

Algebra I

Test I

Directions: Fill in the correct answer on the answer sheet provided.

1. Combine and simplify:

$$[x - 2y - (2x + 3y + 2\{x - 2y\})]$$

2. Evaluate: $xy^2 - 2xy + x(x - 3y)$ when $x = -2$ and $y = 3$.

3. What are the greatest common divisor GCD and least common multiple LCM of: $150x^3y^2$, $12x^2y^3$, and $24xy^3$?

4. Factor the expression completely:

$$2x^4y + 5x^3y^2 - 3x^2y^3$$

5. Solve for x: $2x - 3 = 5x + 6$

6. A grocer mixes two types of nuts to form a 50 pound mixture selling for 60 cents a pound. If type A sells for 85 cents a pound and type B sells for 45 cents a pound, find the amount required of each type.

7. Solve for x and y:

$$2x - 3y = 7$$
$$-5x + 2y = 10$$

8. Simplify the expression:

$$\frac{x^2 - 2xy + y^2}{x^2 - y^2} \div \frac{x^2 - xy}{x^2 - xy - 2y^2}$$

9. Find all solutions to the equation: $2x^2 - x - 6 = 0$.

10. Simplify: $\dfrac{10x^2y^4 - 6x^3y^2 + 8x^2y^3}{2x^2y}$

11. Multiply and simplify: $x^2y(x^2 - 2y^2)(x^4 + 2x^2y^2 + 4y^4)$

12. Find the quotient: $(x^3 - 7xy^2 - 6y^3) \div (x + 2y)$

13. Find all solutions to: $2x - 3 \geq 4x - 7$

West River Math Contest

Algebra I

Test I - Key

Name ______________________ School ______________________

1. $-3x - y$

2. 16

3. GCD = $6xy^2$

   LCM = $600x^3y^3$

4. $x^2y(2x - y)(x + 3y)$

5. -3

6. Type A = 18.75 pounds

   Type B = 31.25 pounds

7. $x = -4$

   $y = -5$

8. $\frac{x - 2y}{x}$

9. $2, -\frac{3}{2}$

10. $5y^3 - 3xy + 4y^2$

11. $x^8y - 8x^2y^7$

12. $x^2 - 2xy - 3y^2$

13. $x \leq 2$

West River Math Contest

Algebra I

Test I - Solutions

1. $[x - 2y - (2x + 3y + 2\{x - 2y\})] =$
$[x - 2y - (2x + 3y + 2x - 4y)] =$
$[x - 2y - (4x - y)] = [x - 2y - 4x + y] = -3x - y$

2. $(-2)(3)^2 - 2(-2)(3) + (-2)(-2 - 3(3)) = -18 + 12 + 22 = 16$

3. $150 = 2 \cdot 3 \cdot 5^2, \quad 12 = 3 \cdot 2^2, \quad 24 = 2^3 \cdot 3$

$GCD = 2 \cdot 3 \cdot xy^2 = 6xy^2$

$LCM = 3 \cdot 5^2 \cdot 2^3 \cdot x^3 \cdot y^3 = 600x^3y^3$

4. $x^2y(2x^2 + 5xy - 3y^2) = x^2y(2x - y)(x + 3y)$

5. $2x - 3 = 5x + 6$
$-9 = 3x$
$x = -3$

6. x = pounds of type A
50 - x = pounds of type B

$85x + 45(50 - x) = 50(60)$
$85x + 2250 - 40x = 3000$
$40x = 750$
$x = 18.75$ pounds of type A
$50 - 18.75 = 31.25$ pounds of type B

7. (1) $2x - 3y = 7$
(2) $-5x + 2y = 10$
(3) $10x - 15y = 35$ (5 times equation (1))
(4) $-10x + 4y = 20$ (2 times equation (2))
(5) $-11y = 55$ (equation (3) + equation (4))
$y = -5$

Substituting into (1) gives

$2x + 15 = 7$
$2x = -8$
$x = -4$

8. $\dfrac{(x - y)^2}{(x - y)(x + y)} \; \dfrac{(x - 2y)(x + y)}{x(x - y)} = \dfrac{x - 2y}{x}$

9. $(2x + 3)(x - 2) = 0$

$2x + 3 = 0 \qquad x - 2 = 0$

$x = -\dfrac{3}{2} \qquad x = 2$

10. $5y^3 - 3xy + 4y^2$

11. $x^2y(x^6 + 2x^4y^2 + 4x^2y^4 - 2x^4y^2 - 4x^2y^4 - 8y^6) =$

$x^2y(x^6 - 8y^6) = x^8y - 8x^2y^7$

12.
$$
\begin{array}{r|l}
 & x^2 - 2xy - 3y^2 \\ \hline
x + 2y & x^3 \qquad\qquad - 7xy^2 - 6y^3 \\
 & \underline{x^3 + 2x^2y} \\
 & \quad - 2x^2y - 7xy^2 \\
 & \quad \underline{- 2x^2y - 4xy^2} \\
 & \qquad\qquad - 3xy^2 - 6y^3 \\
 & \qquad\qquad \underline{- 3xy^2 - 6y^3} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

13. $2x - 3 \geq 4x - 7$

$-2x \geq -4$

$x \leq 2$

West River Math Contest

Algebra I

Test II

Directions: Fill in the correct answer on the answer sheet provided.

1. Find all solutions to the equation: $2x^2 - 6x - 1 = 0$
Simplify the answer as far as possible.

2. Simplify: $\dfrac{\sqrt[3]{2x^2y}}{\sqrt[3]{3xy^2}}$ as far as possible.

3. Simplify the fraction: $\dfrac{2x + 3 + \dfrac{2}{x - 1}}{2 + \dfrac{1}{x - 1}}$

4. The quantity y is inversely proportional to x. If $y = 10$ when $x = \frac{1}{2}$, find y when $x = 35$.

5. Find all solutions to: $\dfrac{2x + 3}{x^2 - 5x + 6} = \dfrac{2}{x - 2} - \dfrac{5}{x - 3}$

6. A farmer rode around his rectangular farm inspecting fences from a jeep. He noticed that the odometer measured the trip at 3.75 miles. If the area of the farm was 560 acres, what were the length of the sides? (1 square mile = 640 acres).

West River Math Contest

Algebra I

Test II - Key

Name ______________________ School ______________________

1. $\frac{3 \pm \sqrt{11}}{2}$

2. $\frac{\sqrt[3]{18xy^2}}{3y}$

3. $x + 1$

4. $\frac{1}{7}$

5. $\frac{1}{5}$

6. length & width 1 mile x $\frac{7}{8}$ mile

West River Math Contest

Algebra I

Test II - Solutions

1. $2x^2 - 6x - 1 = 0$

$$x = \frac{6 \pm \sqrt{36 + 8}}{4} = \frac{6 \pm \sqrt{44}}{4}$$

$$x = \frac{6 \pm 2\sqrt{11}}{4} = \frac{2(3 \pm \sqrt{11})}{4} = \frac{3 \pm \sqrt{11}}{2}$$

2. $$\frac{\sqrt[3]{2x^2y}}{\sqrt[3]{3xy^2}} \frac{\sqrt[3]{9x^2y}}{\sqrt[3]{9x^2y}} = \frac{\sqrt[3]{18x^4y^2}}{3xy} = \frac{x\sqrt[3]{18xy^2}}{3xy} = \frac{\sqrt[3]{18xy^2}}{3y}$$

3. $$\frac{\dfrac{(2x + 3)(x - 1) + 2}{x - 1}}{\dfrac{2(x - 1) + 1}{x - 1}} = \frac{\dfrac{2x^2 + x - 3 + 2}{x - 1}}{\dfrac{2x - 2 + 1}{x - 1}}$$

$$= \frac{\dfrac{2x^2 + x - 1}{x - 1}}{\dfrac{2x - 1}{x - 1}} = \frac{(2x - 1)(x + 1)}{(x - 1)} \frac{(x - 1)}{(2x - 1)} = x + 1$$

4. $y = \frac{k}{x}$

Since $y = 10$ when $x = \frac{1}{2}$, we have $10 = \frac{k}{\frac{1}{2}}$ or $k = 5$.

Now $y = \frac{5}{x}$, so if $x = 35$ we have $y = \frac{5}{35} = \frac{1}{7}$.

5. $\frac{2x + 3}{(x - 3)(x - 2)} = \frac{2}{x - 2} - \frac{5}{x - 3} \qquad x \neq 2, 3$

$2x + 3 = 2(x - 3) - 5(x - 2) \qquad x \neq 2,3$

$2x + 3 = 2x - 6 - 5x + 10$

$5x = 1$

$x = \frac{1}{5}$

6. Length of one side is x miles
thus, the length of the other side is: $\frac{3.75 - 2x}{2}$ miles.

$\frac{560}{640}$ square miles is the size of the farm

$\therefore x\left(\frac{3.75 - 2x}{2}\right) = \frac{560}{640}$

$3.75x - 2x^2 = \frac{7}{4},$

$8x^2 - 15x + 7 = 0$

$(8x - 7)(x - 1) = 0$

$8x - 7 = 0, \quad x - 1 = 0$

$x = \frac{7}{8} \qquad x = 1$

If $x = 1 \quad \frac{3.75 - 2}{2} = \frac{1.75}{2} = \frac{7}{8}$

The dimensions are 1 mile by $\frac{7}{8}$ miles.

West River Math Contest

Geometry

Test I

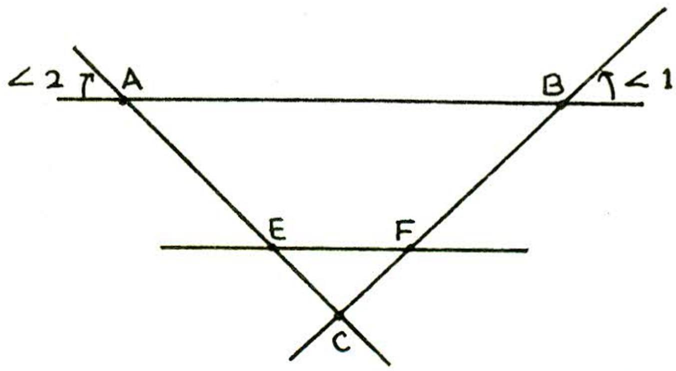
Directions: Write your answer on the answer sheet provided.
Do not convert pi (π) to a decimal.

1. How many vertices are in a hexagon?

2. At 2:00 p.m., a 6 foot tall person casts a shadow of two feet on flat ground. How tall is a tree that casts a shadow of 15 feet at the same time and in the same vicinity as the person?

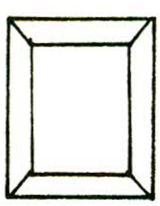
3. Given: $AB \parallel EF$
$AE \perp BF$
$\angle 1 = 50°$

Find: $\angle 2$



4. A bricklayer must lay block for a rectangular basement. To make sure the walls will form a rectangle, and not some other parallelogram, the following measurements are made: from a corner position, a point A is marked six feet from the corner towards an adjacent corner; a point B is marked eight feet from the corner towards the other adjacent corner. If the three corners define a rectangle, then what is the distance from A to B?

5. Wood trim is to surround a window 18" x 3'6". The trim is 2" wide. What is the outside perimeter of the trim? Give answer in feet and inches.

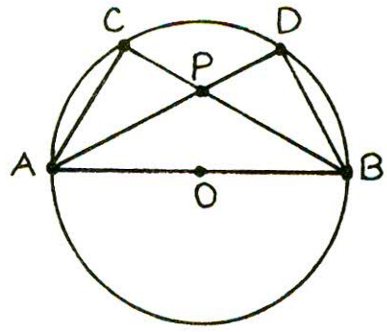


6. Given: a circle with center, O and radius, r.

O is between A and B.
$\triangle ABD$ and $\triangle ABC$ are inscribed.

Points C and D divide $\widehat{AB}$ into 3 equal arcs.

Find: Area of $\triangle ABP$ in terms of r.

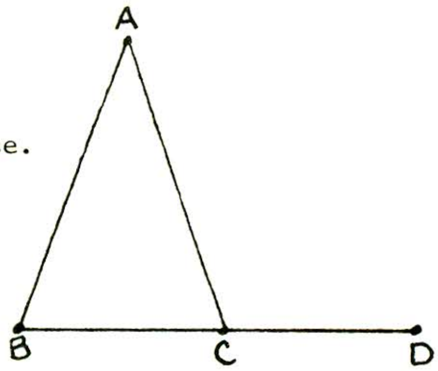


7. Given: a rhombus with sides and one diagonal each having length 12.

Find: the area of the rhombus.

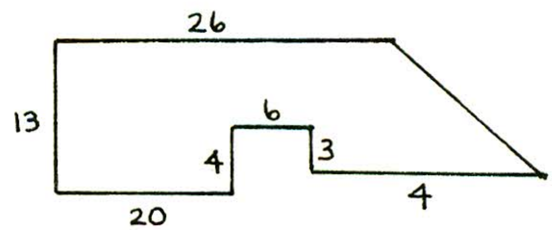
8. Given: $\triangle ABC$ is isosceles with $\overline{BC}$ as the base.
C is between B and D.
$\angle A = 40°$

Find: the measure of $\angle ACD$



9. A rectangle (10" by 24") is circumscribed by a circle of radius R, and has an inscribed circle of radius, r. Find the value of the ratio R/r.

10. Find the area of the geometrical shape having right angles as shown.



11. The circumference of a circle is c. Find the area in terms of c. Simplify your answer.

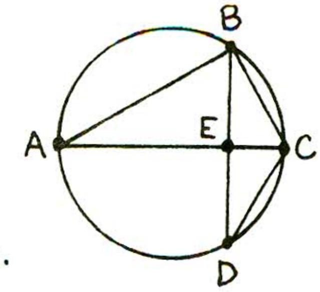
12. An equilateral triangle and a square have the same perimeter, p. What is the ratio of the area of the square to the area of the triangle? Simplify your answer.

13. Given: AC is a diameter of a circle

$BD \perp AC$

$\frac{\overline{AE}}{\overline{EC}} = 5$

Find: the ratio of the areas of $\triangle ABC$ to $\triangle BCD$.



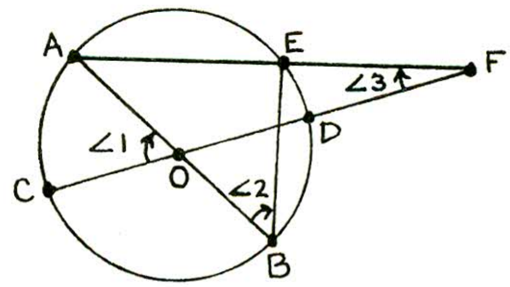
14. Given: a circle with center, O,

chord $\overline{AE}$ and diameter $\overline{CD}$ are extended to intersect at F.

AB is a diameter

$\angle 3 = 20°$, $\angle 1 = 60°$

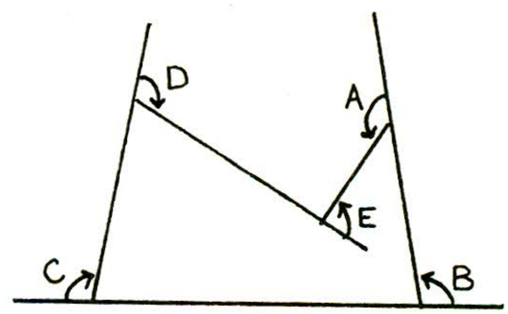
Find: the measure of $\angle 2$



15. Given: an irregular polygon of five sides with exterior angles $\angle A$, $\angle B$, $\angle C$, $\angle D$, all obtuse.

$\angle E = 90°$

Find: $\angle A + \angle B + \angle C + \angle D$



West River Math Contest

Geometry

Test I - Key

Name ______________________ School ______________________

1. 6
2. 45 feet
3. 40°
4. 10 feet
5. 11'4"
6. $\frac{r^2}{\sqrt{3}}$
7. $72\sqrt{3}$
8. 110°
9. $\frac{13}{5}$
10. 338 square units
11. $\frac{C^2}{4\pi}$
12. $\frac{3\sqrt{3}}{4}$
13. 3
14. 50°
15. 450°

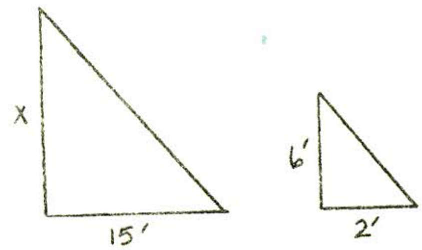
West River Math Contest

Geometry

Test I - Solutions

1. "Hexa" means 6.

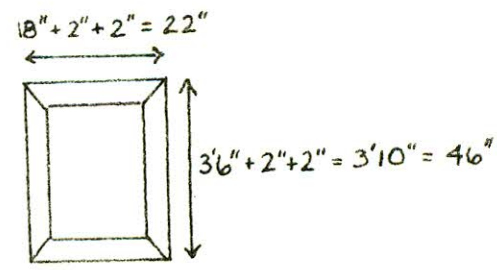
2. The sun casts shadows at the same angle. Assuming the person and the tree stand vertically, the triangles are similar. $\frac{6}{2} = \frac{x}{15}$ or $x = 45'$.



3. $\angle 1 = \angle EFC = 50°$ since $AB \| EF$
$\angle C = 90°$ since $AE \perp BF$
$\angle FEC = 180° - 90° - 50° = 40°$
$\angle FEC = \angle 2 = 40°$ since $AB \| EF$

4. $D = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$ feet, by the Pythagorean Theorem.

5. Perimeter = 2(46) + 2(22) = 136" = 11'4"



6. $\overset{\frown}{AC} = \overset{\frown}{BD} = \frac{1}{3}(180°) = 60°$

Since $\angle CBA$ and $\angle DAB$ are inscribed, their measure is $\frac{1}{2}(60°) = 30°$

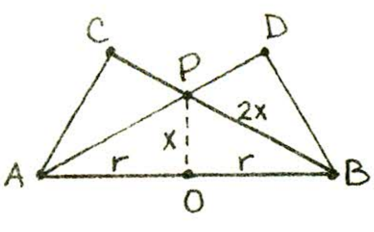
In $\triangle OPB$, $\overline{BP} = 2\overline{PO}$. By the Pythagorean Theorem,

$(2x)^2 = x^2 + r^2$ or

$4x^2 = x^2 + r^2$

$x^2 = \frac{1}{3}r^2$

$x = \frac{r}{\sqrt{3}}$



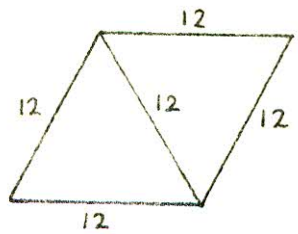
So the area of $\triangle ABP = \frac{1}{2}(2r)\frac{r}{\sqrt{3}} = \frac{r^2}{\sqrt{3}} = \frac{\sqrt{3}r^2}{3}$

7. Double the area of an equilateral triangle since the two triangles are congruent.

Total Area = 2x Area of equilateral triangle

$= 2\,\frac{\sqrt{3}}{4}\cdot 12^2$

$= 72\sqrt{3}$ square units



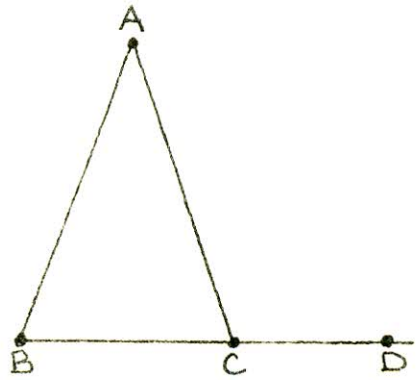
8. $\angle A = 40°$
$\angle B + \angle C = 180° - 40°$
$= 140°$

but $\angle B = \angle C$ since $\triangle ABC$ is isosceles

$\angle C = \frac{1}{2}(140°) = 70°$

$\angle ACD$ is a supplement to $\angle C$

$\angle ACD = 180° - 70° = 110°$



9. Place the small circle in the center.

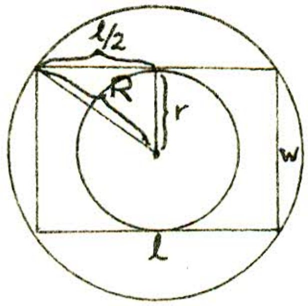
w = 10" implies r = 5"

ℓ = 24" implies $\frac{\ell}{2}$ = 12"

So $R^2 = 12^2 + 5^2$ by the Pythagorean Theorem

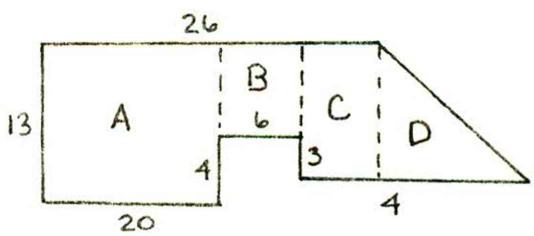
$R = 13$

$\frac{R}{r} = \frac{13}{5}$



10. A is a rectangle 13 x 20 for area of 260.
B is a rectangle 6 x 9 for area of 54.
C is a rectangle 12 x 0 for area of 0.
D is a triangle with base 4 and height 12 for area of 24.

Total area 338 square units.



11. $2\pi r = C$

$r = \frac{C}{2\pi}$

but $A = \pi r^2$; substituting, $A = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{C^2}{4\pi}$

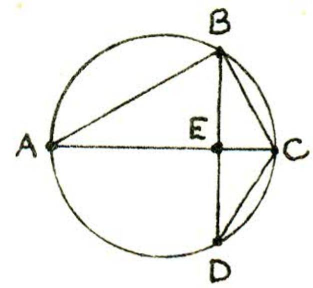
12. The equilateral triangle has each side $\frac{P}{3}$. The formula for the area of an equilateral triangle having side s is $A = \frac{\sqrt{3}}{4} s^2$. For this case we have,

$$A_t = \frac{\sqrt{3}}{4} \frac{P^2}{9} = \frac{\sqrt{3}P^2}{36}.$$

The square has each side $\frac{P}{4}$. The area of a square of side s is $A = s^2$. For this case we have, $A_s = \frac{P^2}{16}$

$$\text{So the ratio } \frac{A_s}{A_t} = \frac{\frac{P^2}{16}}{\frac{\sqrt{3}P^2}{36}} = \frac{36}{\sqrt{3}\ 16} = \frac{3\sqrt{3}}{4}$$

13. $$\frac{\text{Area } \Delta ABC}{\text{Area } \Delta BCD} = \frac{\frac{1}{2}(\overline{AC})(\overline{BE})}{\frac{1}{2}(\overline{BD})(\overline{EC})}$$



But $\overline{BD} = 2\overline{BE}$ since $BD \perp AC$, a diameter.

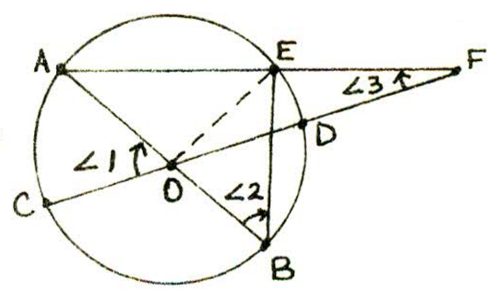
$$\text{So } \frac{\text{Area } \Delta ABC}{\text{Area } \Delta BCD} = \frac{\frac{1}{2}(\overline{AC})(\overline{BE})}{\frac{1}{2}(2)(\overline{BE})(\overline{EC})} = \frac{\overline{AC}}{2\overline{EC}}$$

But $\overline{AC} = \overline{AE} + \overline{EC}$, so ratio $= \frac{\overline{AE} + \overline{EC}}{2\overline{EC}}$

Finally, $\overline{AE} = 5\overline{EC}$ is given,

$$\text{so ratio} = \frac{5\overline{EC} + \overline{EC}}{2\overline{EC}} = \frac{6\overline{EC}}{2\overline{EC}} = 3$$

14. $\overset{\frown}{AC} = \angle 1 = 60°$



$\angle 3 = \frac{1}{2}(\angle 1 - \angle EOD)$ since $\overline{AF}$ and $\overline{CF}$ are intersecting secant lines

Substituting $20° = \frac{1}{2}(60° - \angle EOD)$

or $40° = 60° - \angle EOD$

so $\angle EOD = 20°$

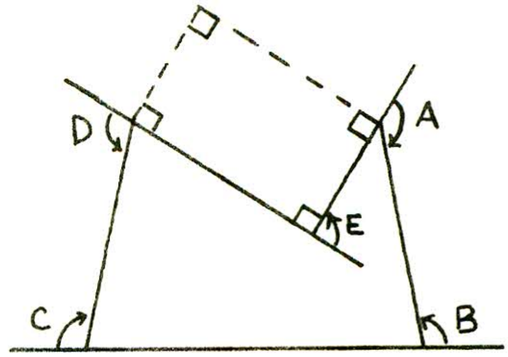
$\overset{\frown}{AE} = \overset{\frown}{CD} - \angle 1 - \angle EOD$

$= 180° - 60° - 20° = 100°$

Finally $\angle 2 = \frac{1}{2}\ \overset{\frown}{AE}$ since they intercept the same arcs

$$\angle 2 = \frac{1}{2}(100°) = 50°$$

15. Insert a rectangle with corners A, E, and D into the right angle. This completes a 5-sided polygon with interior angles adding to $3(180°) = 540°$.



Subtract out the 3 - 90° angles of the rectangle and the interior angles at A, B, C, and D add up to $540° - 3(90°) =$
$540° - 270° = 270°$

But the exterior angles in question are all supplements of the interior angles,

so $(180° - \angle A) + (180° - \angle B) + (180° - \angle C)$
$+ (180° - \angle D) = 270°$

or $\angle A + \angle B + \angle C + \angle D = -270 + 4(180°) = 450°$

West River Math Contest

Geometry

Test II

Directions: Write your answer on the answer sheet provided. Show work for problem #4 on the answer sheet.
Do not convert pi (π) to a decimal.

1. Given trapezoid ABCD

   $BC \perp CD$

   $\overline{BC} = 4$

   $\overline{AD} = 5$

   $\overline{AB} = 9$

   Find: Area of the trapezoid.

2. A triangle with n = 3 sides has a sum of interior angles equal to 180°.
A quadrilaterial with n = 4 sides has a sum of interior angles equal to 360°.
In general what is the sum of the interior angles of an n-sided convex polygon?

3. Given: ΔABC with all angles acute

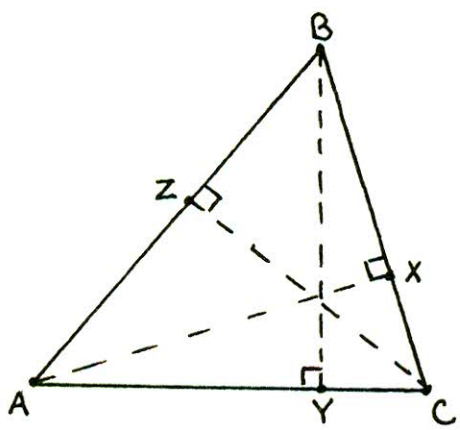
   $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$ are all altitudes

   $\overline{AC} = 8$

   $\overline{BY} = 6$

   $\dfrac{\overline{AX}}{\overline{CZ}} = \dfrac{5}{4}$

   Find: $\dfrac{\overline{AB}}{\overline{BC}}$



4. Four spheres of radius R are placed on the floor with centers at the vertices of a square having sides 2R. A fourth sphere of radius R is placed on top to form a pyramid. What is the height above the floor of the top of the fourth sphere?

   (Show work on the answer sheet).

5. Given: $\triangle ABC \sim \triangle EBD$

In each triangle $\angle B = 90°$

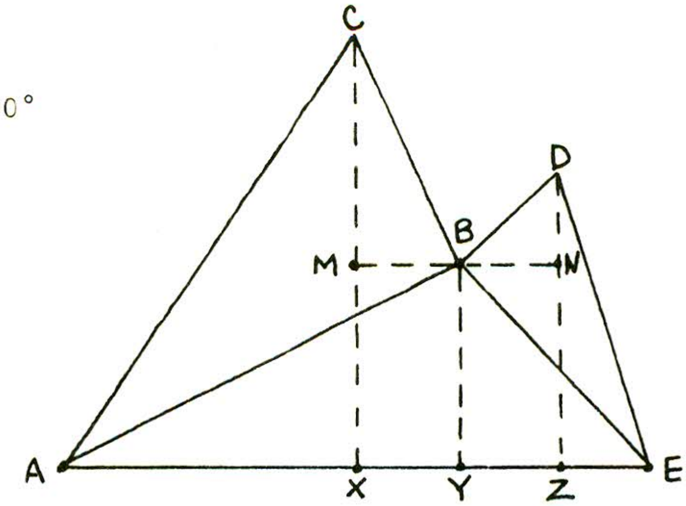
$\overline{DB}$ corresponds to $\overline{CB}$

$\frac{DB}{CB} = \frac{4}{5}$

$\overline{CX}$, $\overline{BY}$ and $\overline{DZ}$ are all perpendicular to AE

$BM \perp CM$, $BN \perp DN$

Find the ratio $\frac{\overline{XY}}{\overline{YZ}}$



West River Math Contest

Geometry

Test II - Key

Name ______________________ School ______________________

1. 42 square units

2. (n - 2)(180°)

3. $\frac{5}{4}$

4. $(2 + \sqrt{2})R$

5. 1

West River Math Contest

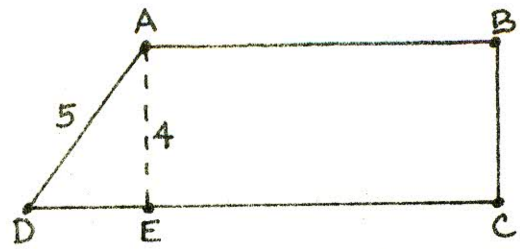
Geometry

Test II - Solutions

1. $\overline{DE} = 3$ from Pythagorean Theorem
   CD = 3 + 9 = 12

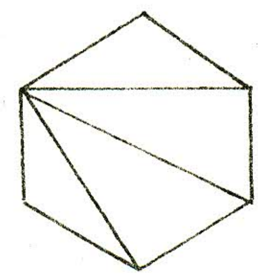
   $$\text{Area} = \frac{12 + 9}{2} \cdot 4$$

   $= 42$ square units



2. A n-sided figure can be divided into n-2 triangles as illustrated. Each triangle has 180° for the sum of its angles.

   The total of the angles for all triangles is $(n - 2)(180°)$.



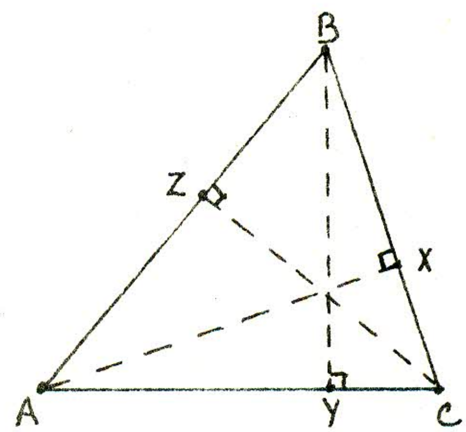
3. $\angle BXA = \angle BZC = 90°$ since AX and CZ are altitudes

   $\triangle BXA \sim \triangle BZC$

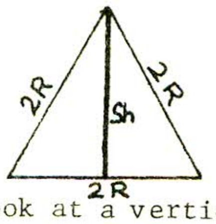
   $$\frac{\overline{AB}}{\overline{BC}} = \frac{\overline{AX}}{\overline{CZ}}$$ by similar triangles

   but $\frac{\overline{AX}}{\overline{CZ}} = \frac{5}{4}$

   So $\frac{\overline{AB}}{\overline{BC}} = \frac{5}{4}$

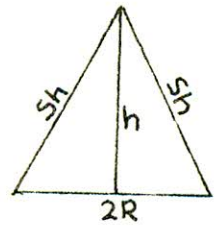


4. The centers of the five spheres form a pyramid. To find the height of the pyramid first look at one side to find the slant height, Sh



$$Sh = \sqrt{4R^2 - R^2}$$
$$= \sqrt{3}\ R$$

Then look at a vertical slice through the center of the pyramid with sides Sh and base 2R.



The vertical height h gives the height of this small pyramid $h = \sqrt{(\sqrt{3}R)^2 - R^2}$

$$= \sqrt{2}\ R$$

Finally, add on the top half of the height of the top sphere and the bottom half of the height of the bottom sphere.
Total height $H = R + R + \sqrt{2}R = (2 + \sqrt{2})R$

5. $\angle BAY = \angle BCM$ since $CB \perp BA$ and $CX \perp AX$
$\triangle BAY \sim \triangle BCM$ because two sets of angles are equal.

So $\dfrac{\overline{BY}}{\overline{AB}} = \dfrac{\overline{BM}}{\overline{CB}}$ or $\overline{BY} = \overline{BM} \cdot \dfrac{\overline{AB}}{\overline{CB}}$

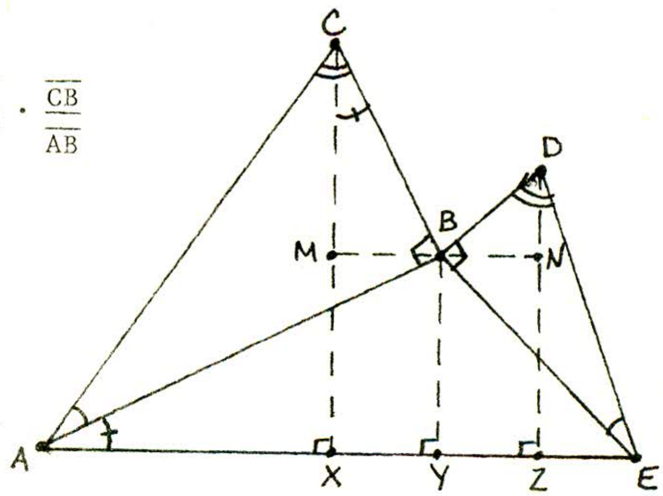
In a like manner,
$\triangle BEY \sim \triangle BDN$

So $\dfrac{\overline{BY}}{\overline{EB}} = \dfrac{\overline{BN}}{\overline{DB}}$ or $\overline{BY} = \overline{BN} \cdot \dfrac{\overline{EB}}{\overline{DB}}$

Equating: $\overline{BM}\ \dfrac{\overline{AB}}{\overline{CB}} = \overline{BN}\ \dfrac{\overline{EB}}{\overline{DB}}$ or $\overline{BM} = \overline{BN}\ \dfrac{\overline{EB}}{\overline{DB}} \cdot \dfrac{\overline{CB}}{\overline{AB}}$

but since $\triangle ABC \sim \triangle EBD$ is given,

$$\frac{\overline{AB}}{\overline{CB}} = \frac{\overline{EB}}{\overline{DB}} \text{ or } \frac{\overline{EB}}{\overline{DB}} \cdot \frac{\overline{CB}}{\overline{AB}} = 1$$



Substituting this would mean $\overline{BM} = \overline{BN}$
Then since $BY \| CX \| DZ$,

$$\overline{XY} = \overline{YZ}$$

So $\dfrac{\overline{XY}}{\overline{YZ}} = 1$

West River Math Contest

Algebra II

Test I

Directions: Fill in the correct answer on the answer sheet provided.

1. Solve for r: $\frac{E}{e} = \frac{R + r}{r}$.

2. Factor completely: $d^2n - 4n - 8d^2 + 32$.

3. Find all values of x for which $2x^3 - 3x = 5x^2$.

4. Express as a single fraction and reduce, if possible:

$$\frac{1}{n + 5} + \frac{2n}{25 - n^2}.$$

5. Simplify: $\dfrac{n + 4 + \frac{3}{n}}{1 - \frac{1}{n^2}}$.

6. How many pounds of 44-cent candy and how many pounds of 60-cent candy must be mixed to make 30 pounds of 54-cent candy?

7. Write in simplest form without negative exponents:

$$(2 + x)^{-2}(1 + 2x^{-1}).$$

8. Write as a single radical, reduced as much as possible:

$$\sqrt{n} \cdot \sqrt[3]{n^2}.$$

9. While traveling with the wind, a plane flew the 300 miles between two airports in 40 minutes. It returned against the wind in 45 minutes. Find the rate of the wind in miles per hour.

10. Find the x-value of the system: $\begin{Bmatrix} x + y + z = 4 \\ x + y - z = 12 \\ x - y + 3z = -14 \end{Bmatrix}$.

11. Simplify: $\dfrac{\sqrt{ab^5} + 4\sqrt{a^5b^3} - b\sqrt{ab^3}}{\sqrt{a^3b^3}}$.

12. Find all values of x for which $x - \sqrt{3x - 2} = 2$.

13. If $f(x) = \dfrac{x^2 - a^2}{x}$, find $f(x - a)$.

14. Write in simplest form without negative exponents:

$$\frac{b^3\left(a^{-\frac{1}{9}}\right)^{-3}}{\left(a^2b^{-3}\right)^{\frac{2}{3}}}.$$

15. B varies inversely as the square of d. If B = 10 when d = 2, find B when d = 4.

West River Math Contest

Algebra II

Test I - Key

Name ______________________ School ______________________

1. $r = \frac{Re}{E - e}$

2. $(d + 2)(d - 2)(n - 8)$

3. $x = 0, -\frac{1}{2}, 3$

4. $\frac{1}{5 - n}$

5. $\frac{n(n + 3)}{n - 1}$

6. $\frac{45}{4}$ lbs. of 44¢ candy
   $\frac{75}{4}$ lbs. of 60¢ candy

7. $\frac{1}{x(x + 2)}$

8. $n\sqrt[6]{n}$

9. 25 mph wind

10. $x = 3$

11. $4a$

12. $x = 6$

13. $\frac{x^2 - 2ax}{x - a}$

14. $\frac{b^5}{a}$

15. $B = \frac{5}{2}$

West River Math Contest

Algebra II

Test I - Solutions

1. $\frac{E}{e} = \frac{R + r}{r}$

$Er = (R + r)e$
$Er = Re + re$
$Er - re = Re$
$r(E - e) = Re$
$r = \frac{Re}{E - e}$

2. $d^2n - 4n - 8d^2 + 32$

$= n(d^2 - 4) - 8(d^2 - 4)$
$= (d^2 - 4)(n - 8)$
$= (d + 2)(d - 2)(n - 8)$

3. $2x^3 - 3x = 5x^2$

$2x^3 - 5x^2 - 3x = 0$
$x(2x^2 - 5x - 3) = 0$
$x(2x + 1)(x - 3) = 0$
$x = 0,\ x = -\frac{1}{2},\ x = 3$

4. $\frac{1}{n + 5} + \frac{2n}{25 - n^2}$

$= \frac{1}{n + 5} - \frac{2n}{n^2 - 25}$

$= \frac{1}{n + 5} - \frac{2n}{(n + 5)(n - 5)}$

$= \frac{n - 5 - 2n}{(n + 5)(n - 5)}$

$= \frac{-(5 + n)}{(n + 5)(n - 5)}$

$= \frac{-1}{n - 5}$

$= \frac{1}{5 - n}$

5. $\dfrac{n + 4 + \frac{3}{n}}{1 - \frac{1}{n^2}}$

$= \frac{n^3 + 4n^2 + 3n}{n^2 - 1}$

$= \frac{n(n + 3)(n + 1)}{(n + 1)(n - 1)}$

$= \frac{n(n + 3)}{n - 1}$

6. Let x = lbs. of 44-cent candy. Then 30 - x = lbs. of 60-cent candy.

$.44x + .60(30 - x) = .54(30)$
$44x + 60(30 - x) = 54(30)$
$-16x = 54(30) - 60(30)$
$-16x = -6(30)$
$x = \frac{6(30)}{16}$
$x = \frac{45}{4}$ lbs. of 44-cent candy
$30 - x = \frac{75}{4}$ lbs. of 60-cent candy

7. $(2 + x)^{-2}(1 + 2x^{-1})$

$= \dfrac{1 + \frac{2}{x}}{(x + 2)^2}$

$= \frac{x + 2}{x(x + 2)^2}$

$= \frac{1}{x(x + 2)}$

8. $\sqrt{n} \cdot \sqrt[3]{n^2}$

$= n^{1/2} \cdot n^{2/3}$

$= n^{7/6}$

$= \sqrt[6]{n^7}$

$= n\sqrt[6]{n}$

9. Let x = airplane rate in still air.

y = wind rate

Then $300 = (x + y)\,\frac{40}{60}$

$300 = (x - y)\,\frac{45}{60}$

$900 = 2x + 2y$
$1200 = 3x - 3y$

$450 = x + y$
$400 = x - y$

$50 = 2y$
$y = 25$ mph wind

10. $\begin{cases} x + y + z = 4 \\ x + y - z = 12 \\ x - y + 3z = -14 \end{cases}$

$\begin{cases} 2x + 2y = 16 \\ 4x + 2y = 22 \end{cases}$

$-2x = -6$
$x = 3$

11. $\dfrac{\sqrt{ab^5} + 4\sqrt{a^5b^3} - b\sqrt{ab^3}}{\sqrt{a^3b^3}}$

$= \dfrac{b^2\sqrt{ab} + 4a^2b\sqrt{ab} - b^2\sqrt{ab}}{ab\sqrt{ab}}$

$= \dfrac{4a^2b}{ab}$

$= 4a$

12. $x - \sqrt{3x - 2} = 2$

$x - 2 = \sqrt{3x - 2}$
$x^2 - 4x + 4 = 3x - 2$
$x^2 - 7x + 6 = 0$
$(x - 6)(x - 1) = 0$
$x = 6, \quad x = 1$
But $x = 1$ is extraneous
$x = 6$

13. $f(x) = \dfrac{x^2 - a^2}{x}$

$f(x - a) = \dfrac{(x - a)^2 - a^2}{x - a}$

$= \dfrac{x^2 - 2ax}{x - a}$

14. $\dfrac{b^3\left(a^{-\frac{1}{9}}\right)^{-3}}{\left(a^2b^{-3}\right)^{2/3}}$

$= \dfrac{b^3a^{1/3}}{a^{4/3}b^{-2}}$

$= \dfrac{b^5}{a}$

15. $B = \dfrac{k}{d^2}$

$10 = \dfrac{k}{4}$

$k = 40$

$B = \dfrac{40}{d^2}$

$B = \dfrac{40}{16}$

$B = \dfrac{5}{2}$

West River Math Contest

Algebra II

Test II

Directions: Fill in the correct answer on the answer sheet provided.

1. Solve for t: $h = vt - 16t^2$.

2. Solve for x: $2x^{-2} + 7x^{-1} = \frac{1}{8}$.

3. Solve for n: $I = \dfrac{E}{R + \frac{r}{n}}$.

4. Write in simplest form without negative exponents:

$$\frac{(x^{-2} - y^{-2})^{-1}}{(x^{-1} - y^{-1})^{-2}}.$$

5. The sum of the digits of a two-digit number is 11. If 27 is subtracted from the number, the digits of the number are reversed. What is the number?

   Show your work on the answer sheet.

West River Math Contest

Algebra II

Test II - Key

Name ______________________ School ______________________

1. $t = \frac{v \pm \sqrt{v^2 - 64h}}{32}$

2. $x = \frac{4}{-7 \pm 5\sqrt{2}}$

3. $n = \frac{Ir}{E - IR}$

4. $\frac{y - x}{y + x}$

5. 74

West River Math Contest

Algebra II

Test II - Solutions

1. $h = vt - 16t^2$

$16t^2 - vt + h = 0$

$t = \frac{v \pm \sqrt{v^2 - 64h}}{32}$

2. $2x^{-2} + 7x^{-1} = \frac{1}{8}$

Let $x^{-1} = y$

$2y^2 + 7y - \frac{1}{8} = 0$

$y = \frac{-7 \pm \sqrt{49 + 1}}{4}$

$= \frac{-7 \pm 5\sqrt{2}}{4}$

$x = \frac{4}{-7 \pm 5\sqrt{2}}$

3. $I = \frac{E}{R + \frac{r}{n}}$

$I = \frac{En}{Rn + r}$

$IRn + Ir = En$

$(IR - E)n = -Ir$

$n = \frac{-Ir}{IR - E}$

$= \frac{Ir}{E - IR}$

4. $\frac{(x^{-2} - y^{-2})^{-1}}{(x^{-1} - y^{-1})^{-2}}$

$= \frac{(x^{-1} - y^{-1})^2}{x^{-2} - y^{-2}}$

$= \frac{\left(\frac{1}{x} - \frac{1}{y}\right)^2}{\frac{1}{x^2} - \frac{1}{y^2}}$

$= \frac{\left(\frac{y - x}{xy}\right)^2}{\frac{y^2 - x^2}{x^2y^2}}$

$= \frac{(y - x)^2}{(y + x)(y - x)}$

$= \frac{y - x}{y + x}$

5. Let u = units digit
t = tens digit

then, $t + u = 11$

Let N = the number

then $N = 10t + u$

$N - 27 = 10u + t$

$10t + u - 27 = 10u + t$

$9t - 9u = 27$

$\begin{cases} t - u = 3 \\ t + u = 11 \end{cases}$

$2t = 14$

$t = 7$

$u = 4$

$N = 74$

West River Math Contest

Advanced Mathematics

Test I

Directions: Circle the correct answer on the answer sheet provided.

1. Express: $\frac{\sqrt[3]{x^2}}{\sqrt[5]{x^3}}$ as a single radical.

(a) $\sqrt[5]{x^2}$ (b) $\sqrt[9]{x^{10}}$ (c) $\sqrt[15]{x}$ (d) $\sqrt[15]{x^6}$ (e) $\sqrt[8]{x^5}$

2. Find the set A where $A = \{x \mid |x - 1| < 1\} \cap \{x \mid 1 - 2x \leq -2\}$.

(a) $[-2, -1)$ (b) $[\frac{1}{2}, 2]$ (c) $[\frac{3}{2}, 2)$ (d) $(-1, 1)$

(e) The intersection is empty.

3. Given: $2y + 3x = 5$; if we increase y by the amount a, find the change in x.

(a) $\frac{2}{3}a$ (b) $-\frac{2}{3}a$ (c) $\frac{3}{2}a$ (d) $-\frac{3}{2}a$ (e) a

4. Solve for x and y: $2^x \cdot 3^y = 11$
$3^x \cdot 2^y = 13$

(a) $x = 1$, $y = 2$ (b) $x = 2$, $y = 2$ (c) $x = \frac{\log 11 \log 2 - \log 3 \log 13}{\log^2 2 - \log^2 3}$

$y = \frac{\log 2 \log 13 - \log 3 \log 11}{\log^2 2 - \log^2 3}$

(d) $x = \frac{\log 13 - \log 16}{\log^2 2 - \log^2 3}$ (e) None of these

$y = \frac{\log 15 - \log 14}{\log^2 2 - \log^2 3}$

5. To which of the following is $\sec^2 x + \csc^2 x$ equal?

(a) $\sec^2 x \csc^2 x$ (b) $\tan^2 x \cot^2 x$ (c) 1 (d) $\tan^2 x + \cot^2 x$

(e) None of these

6. Solve for all values of θ on the range $0 \leq \theta < 360°$;

$$\sin 2\theta + \sin\theta = 0.$$

(a) $\theta = 0°, 90°, 180°, 270°$

(b) $\theta = 0°, 120°, 180°, 240°$

(c) $\theta = 0°, 180°$

(d) $\theta = 0°, 120°, 240°$

(e) None of these

7. Express as a single logarithm: $\log_{100}3 + \log_{10}5$.

(a) $\log_{10}5\sqrt{3}$ (b) $\frac{1}{2}\log_{10}15$ (c) $\log_{10}15$ (d) impossible

8. If n is a positive integer greater than 2, what is the value of:

$$\frac{2!n!}{6!(n-1)!}$$

(a) $\frac{n}{6}$ (b) $\frac{1}{360}$ (c) $\frac{1}{3(n-1)}$ (d) $\frac{n}{360}$ (e) None of these

9. Which of the following most accurately represents the phrase, "x is inversely proportional to A and directly proportional to the cube of B".

(a) $x = \frac{B^3}{A}$ (b) $x = \frac{A}{B^3}$ (c) $x = \frac{AC}{B^3}$ (d) $x = \frac{B^3C}{A}$

(e) $x = AB^3$

West River Math Contest

Advanced Mathematics

Test I - Key

Name ______________________ School ______________________

1. c
2. c
3. b
4. c
5. a
6. b
7. a
8. d
9. d
10. b
11. a
12. b
13. c
14. b
15. c

West River Math Contest

Advanced Mathematics

Test I - Solutions

1. $\dfrac{\sqrt[3]{x^2}}{\sqrt[5]{x^3}} = \dfrac{x^{2/3}}{x^{3/5}} = x^{2/3 - 3/5} = x^{10/15 - 9/15} = x^{1/15} = \sqrt[15]{x}$

2. $|x - 1| < 1$ implies: $-1 < x - 1 < 1$ or $0 < x < 2$

$1 - 2x \leq -2$ implies: $-2x \leq -3$ or $x \geq \frac{3}{2}$

The intersection is $A = \{x | \frac{3}{2} \leq x < 2\}$ or $[\frac{3}{2}, 2)$.

3. Given: $2y + 3x = 5$; $x = -\frac{2}{3}y + \frac{5}{3}$. If we increase y by amount a,

$x_1 = -\frac{2}{3}(y + a) + \frac{5}{3}$, therefore x increases by the amount $-\frac{2}{3}a$.

4. $2^x \cdot 3^y = 11$, $3^x \cdot 2^y = 13$ give: $x\log 2 + y\log 3 = \log 11$, $x\log 3 + y\log 2 = \log 13$.

and solving as usual gives:

$$x = \frac{\log 11 \log 2 - \log 3 \log 13}{\log^2 2 - \log^2 3} \qquad y = \frac{\log 2 \log 13 - \log 3 \log 11}{\log^2 2 - \log^2 3}$$

5. $$\sec^2 x + \csc^2 x = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x \cdot \sin^2 x}$$

$$= \frac{1}{\cos^2 x \sin^2 x} = \sec^2 x \csc^2 x$$

6. $\sin 2\theta + \sin\theta = 0$

$2\sin\theta\cos\theta + \sin\theta = 0$

$\sin\theta(2\cos\theta + 1) = 0$

$\sin\theta = 0$ or $\cos\theta = -\frac{1}{2}$

$\theta = 0°, 180°, 120°, 240°$

7. If $n = \log_{100}3$, then $100^n = 3$, $10^{2n} = 3$, $2n = \log_{10}3$

$n = \frac{1}{2}\log_{10}3$, $n = \log_{10}\sqrt{3}$, Thus,

$\log_{100}3 + \log_{10}5 = \log_{10}\sqrt{3} + \log_{10}5 = \log_{10}5\sqrt{3}$

8. $\frac{2!n!}{6!(n-1)!} = \frac{2\cdot 1\cdot n\cdot (n-1)!}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot (n-1)!} = \frac{n}{360}$

9. x is proportional to $\frac{B^3}{A}$. Let C be the proportionality constant and $x = \frac{B^3C}{A}$.

10. $V = \frac{4}{3}\pi R^3$, $S = 4\pi R^2$, so $V = \frac{1}{3}(4\pi R^2)R = \frac{SR}{3}$

11. From 8 letters, choose 3, or $\binom{8}{3} = \frac{8!}{3!5!} = \frac{8\cdot 7\cdot 6\cdot 5!}{3\cdot 2\cdot 5!} = 56$

12. Simplify: $(i^4 - 2i^3 + 3i^2 - 5i + 4)^2$

$i^4 = 1$, $i^3 = -i$, so

$[1 - 2(-i) + 3(-1) - 5i + 4]^2 = (2 - 3i)^2 = 4 - 12i - 9 = -5 - 12i$

13. The general quadratic form for $xy + x = 10$ is

$$0x^2 + 1xy + 0y^2 + x + 0y - 10 = 0$$

Thus, $A = 0$, $B = 1$, $C = 0$ and the discriminant $B^2 - 4AC = 1 > 0$ and the quadratic is a hyperbola.

14. The coodinates of the center of the circle $x^2 + y^2 + 6x - 4y - 11 = 0$ are found by completing the square;

$(x^2 + 6x + 9) + (y^2 - 4y + 4) = 11 + 13$ or

$(x + 3)^2 + (y - 2)^2 = 24$ and the center is at $(-3, 2)$.

15. Let x = cost/bushel. The selling price for 16 bushels is 1.5x. The selling price for 4 bushels is x and the selling price for 5 bushels is 0. Thus,

$$16(1.5x) + 4(x) + 5(0) - 25x = 6$$

$$24x + 4x - 25x = 3x = 6 \text{ or } x = \$2/\text{bushel}$$

West River Math Contest

Advanced Mathematics

Test II

Directions: Circle the correct answer on the answer sheet provided.

1. Evaluate: $\text{Arc cos}\left(\frac{1}{2}\right) - \text{Arc cos}\left(-\frac{1}{2}\right)$

(a) $-\frac{\pi}{3}$ radians (b) $-\frac{\pi}{6}$ radians (c) 0 (d) $\frac{\pi}{6}$ radians

(e) $\frac{\pi}{3}$ radians

2. The mean proportional between $\frac{x - 2}{x + 2}$ and $(x^2 - 4)$ is:

(a) $(x + 2)^2$ (b) $\pm(x - 2)$ (c) $x^2 + 4$ (d) None of these

3. The fifth term in the sequence $\frac{3}{2}, 1, \frac{7}{10}, \frac{9}{17}$ is:

(a) $\frac{11}{24}$ (b) $\frac{11}{26}$ (c) $\frac{12}{25}$ (d) None of these

4. Given: $P(x) = (1 + i)x - 1 + i$ with $i = \sqrt{-1}$, what is the value of $P(i)$?

(a) -2 (b) 0 (c) $2i - 2$ (d) $2i$ (e) None of these

5. If $\begin{vmatrix} x & -2 & 1 \\ -1 & x & 1 \\ 2 & -2 & 1 \end{vmatrix} = 0$, then what is x?

(a) ± 2 (b) $\pm 2\sqrt{2}$ (c) 0 (d) None of these

6. In the expansion of $(x - 2y)^7$, what is the coefficient of the term containing the product x^4y^3?

(a) 252 (b) -280 (c) -540 (d) None of these

7. If $F(x) = 3x + 2$ and $G(x) = 6x - 5$, what is: $F(3 + G(2))$?

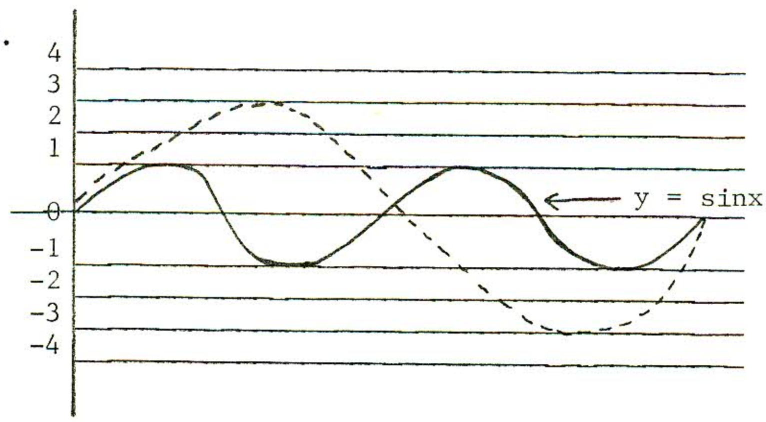
(a) 8 (b) 35 (c) 32 (d) 47 (e) None of these

8. Write the equation of a line perpendicular to the line $x - 3y + 6 = 0$ and passing through the point (1, 0).

(a) $y = 3x + 1$ (b) $x - y - 2 = 0$ (c) $y = -3x + 3$

(d) $y = \frac{x}{3}$ (e) None of these

9.



Which of the following best describes the dotted curve?

(a) $y = 3\sin\frac{x}{2}$
(b) $y = 3\sin x$
(c) $y = 3\cos 2x$
(d) $y = 3\sin 4x$
(e) $y = \sin x \cos x$

10. What distance will a rubber ball travel before coming to rest if it is dropped from a height of 6 ft. and after each fall it rebounds two thirds of the distance it falls?

(a) 24 ft. (b) 30 ft. (c) 18 ft. (d) 32 ft. (e) None of these

West River Math Contest

Advanced Mathematics

Test II - Key

Name ______________________ School ______________________

1. a

2. b

c. b

4. c

5. a

6. b

7. c

8. c

9. a

10. b

West River Math Contest

Advanced Mathematics

Test II - Solutions

1. Let $\theta_1 = \text{Arc}\cos\frac{1}{2}$, then $\cos\theta_1 = \frac{1}{2}$ and for principal values $(-\pi \leqq \theta \leqq \pi)$, $\theta_1 = \frac{\pi}{3}$

Let $\theta_2 = \text{Arc}\cos\left(-\frac{1}{2}\right)$, then $\cos\theta_2 = -\frac{1}{2}$ and $\theta_2 = \frac{2\pi}{3}$

Then $\theta_1 - \theta_2 = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$

2. By definition the mean proprotional between $\frac{x-2}{x+2}$ and $x^2 - 4$ is:

$$\frac{\frac{x-2}{x+2}}{y} = \frac{y}{x^2-4} \quad \text{or} \quad (x^2-4)\frac{x-2}{x+2} = (x-2)^2 = y^2$$

Thus, $y = \pm(x-2)$

3. The denominator is following the pattern,

$2, 5, 10, 17, \ldots, n^2 + 1, \ldots$

and the numerator is following the pattern,

$3, 5, 7, 9, \ldots, 2n + 1, \ldots$

Thus, the fifth term is $\frac{2n+1}{n^2+1}$; for $n = 5$ or $\frac{2(5)+1}{5^2+1} = \frac{11}{26}$

4. $P(x) = (1 + i)x - 1 + i$
$P(i) = (1 + i)i - 1 + i = i - 1 - 1 + i = 2i - 2$

5. Expanding about the 3rd column, we get $\begin{vmatrix} -1 & x \\ 2 & -2 \end{vmatrix} - \begin{vmatrix} x & -2 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} x & -2 \\ -1 & x \end{vmatrix}$

$= (2 - 2x) - (-2x + 4) + (x^2 - 2)$

$= x^2 - 4 = 0$ or $x = \pm 2$

6. By the binomial theorem the coefficient is:

$$(-2)^3\binom{7}{3} = -8\,\frac{7!}{4!3!} = -280$$

7. $G(2) = 7$, $F(3 + G(2)) = F(10) = 3(10) + 2 = 32$

8. $x - 3y + 6 = 0$ gives $y = \frac{1}{3}x + 2$ which has a slope of $\frac{1}{3}$.

A line perpendicular has a slope of -3. The equation is $y = -3x + b$ which must satisfy $x = 1$, $y = 0$.

Thus, $0 = -3 + b$ or $b = 3$. Thus the equation is $y = -3x + 3$.

9. The amplitude is $A = 3$. The period is $4\pi = \frac{2\pi}{B}$ or $B = \frac{1}{2}$

Thus the best is $y = 3\sin\frac{1}{2}x$.

10. $6 + 2\left(\frac{2}{3}\right)6 + 2\left(\frac{4}{9}\right)6 + 2\left(\frac{8}{27}\right)6 + \dots$

$$= 6\left[1 + 2\left(\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots\right)\right]$$

$$= 6\left[1 + 2\left(\frac{\frac{2}{3}}{1 - \frac{2}{3}}\right)\right] = 6(1 + 4) = 30 \text{ ft.}$$