50th Annual West River Math Contest

May 15, 2000

South Dakota
School of Mines & Technology

West River Math Contest
ALGEBRA I TEST I
2000

Fill in the correct answer on the answer sheet

1. Combine and simplify: $40y-16-2\{y-3[2x-2(5y+x-2)]\}$

2. Perform the following operations and simplify: $4\cdot 3^2-6\div 3$

3. Multiply and simplify: $(x^2-2x)(2x^3+x^2-2x)$

4. Simplify: $\dfrac{x^2-4x+3}{x^2-2x-3}\cdot\dfrac{x^3-x}{x^2-2x+1}$

5. Simplify: $\dfrac{2}{z+2}+\dfrac{z}{z^2-4}-\dfrac{3z-4}{z^2}$

6. When dividing $x^4-x^3+2x^2-4x+1$ by x^2+2x-3, find the quotient and remainder.

7. Determine which of the points lie on the graph of $y=x^2-2x+3$.

a. (2,3)
b. (-1,0)
c. (-3,6)

8. Solve for x: $2x-4=5x-7$

9. Find all solutions to the equation $2x^2+x-6=0$.

10. Jane has 37 dollars more than Bob. If the amount of money that Bob has is doubled, he will have 5 dollars more than Jane. How much money does each person have?

11. Simplify $\dfrac{x-\frac{1}{x}}{1+\frac{1}{x}}$

12. Simplify $\left(\dfrac{3x^2y^3}{2x^3y}\right)^2$

13. Simplify $(2m^3s^2)^2\left(3m^3s\right)^3$

ALGEBRA I
TEST I SOLUTIONS

1. $40y-16-2\{y-3[2x-10y-2x+4]\}=40y-16-2\{y+30y-12\}$
$=40y-16-62y+24=-22y+8$

2. $4\cdot 9-2=36-2=34$

3. $2x^5+x^4-2x^3-4x^4-2x^3+4x^2=2x^5-3x^4-4x^3+4x^2$

4. $\dfrac{(x-3)(x-1)}{(x-3)(x+1)}\cdot\dfrac{x(x-1)(x+1)}{(x-1)^2}=x$

5. $\dfrac{2}{z+2}+\dfrac{z}{(z-2)(z+2)}-\dfrac{3z-4}{z^2}=\dfrac{2z^3-4z^2+z^3-3z^3+12z+4z^2-16}{z^2(z-2)(z+2)}$
$=\dfrac{12z-16}{z^2(z^2-4)}$

6.
$$\begin{array}{r}
x^2-3x+11 \\
x^2+2x-3\,\big)\,x^4-x^3+2x^2-4x+1 \\
\underline{x^4+2x^3-3x^2} \\
-3x^3+5x^2-4x \\
\underline{-3x^3-6x^2+9x} \\
11x^2-13x+1 \\
\underline{11x^2+22x-33} \\
-35x+34
\end{array}$$

$Quotient=x^2-3x+11$
$Remainder=-35x+34$

7. a. $x=2$ then $y=3=3$ -- yes
   b. $x=-1$ then $y=6\neq 0$ -- no
   c. $x=-3$ then $y=18\neq 6$ -- no

8. $-3x=-3$
   $x=1$

9. $(x+2)(2x-3)=0$
   $x=-2$ and $x=3/2$

10. Bob has x dollars

Jane has $x+37$ dollars

$2x-5=x+37$

$2x=x+42$

$x=42$ dollars

$x+37=79$ dollars

11. $\dfrac{\frac{x^2-1}{x}}{\frac{x+1}{x}}=\dfrac{(x-1)(x+1)}{x}\cdot\dfrac{x}{x+1}=x-1$

12. $\dfrac{9x^4y^6}{4x^6y^2}=\dfrac{9y^4}{4x^2}$

13. $4m^6s^4\cdot 27\cdot m^9s^3=108m^{15}s^7$

West River Math Contest
ALGEBRA I TEST II
2000

Fill in the correct answer on the answer sheet

1. Solve for y: $3y - 2x = 4 - ay$

2. Factor completely: $12a^3b^4 + 10a^4b^3 - 8a^5b^2$

3. Walnuts costing \$3.50 per pound are mixed with peanuts costing \$1.50 per pound to produce a 10 pound mixture worth \$1.80 per pound. Find the number of pounds of each type that must be mixed together.

4. Combine and simplify: $\frac{x}{x^2 - 1} - \frac{x+3}{x^2 - 5x + 4} + \frac{8}{x^2 - 3x - 4}$

5. Combine and simplify: $\frac{48a^3b^2}{25c^2} \cdot \frac{15c^3}{16a^2b} \div \frac{81ac^2}{35b^3}$

6. Combine and simplify: $\left(\frac{9a^3b^2}{6c^3d^4}\right)^3 \left(\frac{10acd^3}{15a^3b^2d}\right)^4$

7. Combine and simplify : $\frac{2}{x-2} - \frac{x+2}{x^2 - 2x} - \frac{x-2}{x^2}$

ALGEBRA 1 TEST II
SOLUTIONS

1. $3y + ay = 2x + 4$

$y = \dfrac{2x+4}{a+3}$

2. $2a^3b^2(6b^2 + 5ab - 4a^2) = 2a^3b^2(2b - a)(3b + 4a)$

3. x = number of pounds of walnuts

   $10 - x$ = number of pounds of peanuts

$3.5x + 1.5(10 - x) = 18$

$2x + 15 = 18$

$x = 1.5$ pounds of walnuts

$10 - x = 8.5$ pounds of peanuts

4. $\dfrac{x}{(x-1)(x+1)} - \dfrac{x+3}{(x-4)(x-1)} + \dfrac{8}{(x-4)(x+1)} = \dfrac{x^2 - 4x - x^2 - 4x - 3 + 8x - 8}{(x-1)(x+1)(x-4)} = \dfrac{-11}{(x-1)(x+1)(x-4)}$

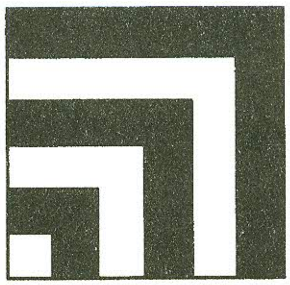
5. $\dfrac{48a^3b^2}{25c^2}\dfrac{15c^3}{16a^2b}\dfrac{35b^3}{81ac^2} = \dfrac{7b^4}{9c}$

6. $\dfrac{3^6a^9b^6}{2^33^3c^9d^{12}}\dfrac{2^45^4a^4c^4d^{12}}{3^45^4a^{12}b^8d^4} = \dfrac{2a}{3b^2c^5d^4}$

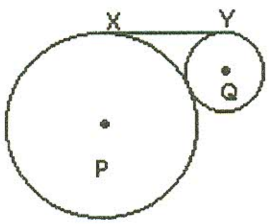
7. $\dfrac{2}{x-2} - \dfrac{x+2}{x(x-2)} - \dfrac{x-2}{x^2} = \dfrac{2x^2 - x^2 - 2x - x^2 + 4x - 4}{x^2(x-2)}$

$= \dfrac{2(x-2)}{x^2(x-2)} = \dfrac{2}{x^2}$

West River Math Contest
Geometry Test I
2000



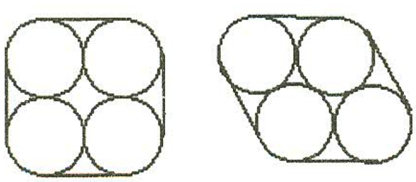
1. What fraction of this square region is shaded? The stripes are equal in width.

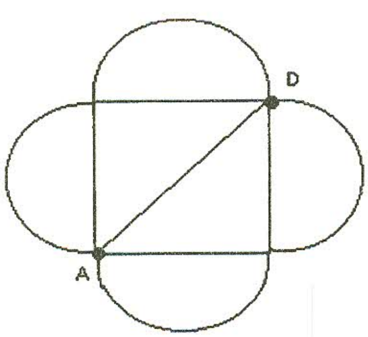


2. Circles P and Q are tangent and have radii of 9 and 4, respectively. Find the length of the common tangent $\overline{XY}$.

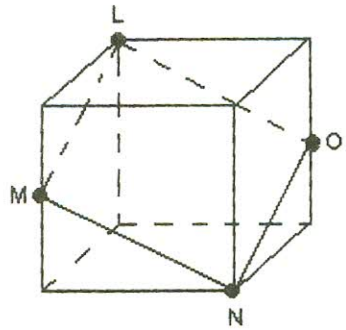
3. A 2-inch cube of silver weighs 3 pounds and is worth $200. How much is a 3-inch cube of silver worth?

4. Four circles of radius 2 are arranged in a "square" and enclosed in the shortest possible perimeter. Then they are arranged in a "parallelogram." What is the difference in the 2 perimeters?



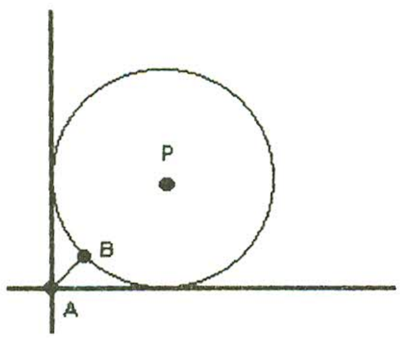


5. A square has semicircles of equal radii attached to each side. If the outside perimeter of this figure is 3π inches, find length $\overline{AD}$.



6. A cube with edge of 5 is cut by a plane to create quadrilateral LMNO, where M and O are midpoints of 2 edges of the cube. Find the area of LMNO.

7. If circle P has a radius of 1 unit, find length $\overline{AB}$.



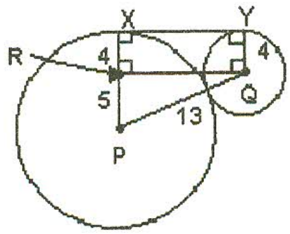
8. Two congruent hexagons overlap so that a vertex of one lies on the center of the other. Intersecting sides are perpendicular. Find the difference in the non-overlapping areas.

9. An equilateral triangle of area $25\sqrt{3}$ square units has the same perimeter as a regular hexagon. Find the length of a side of the hexagon.

10. A hollow piece of cylindrical pipe has an outside radius of 3 inches and an inside radius of 2 inches. If the pipe is 2 feet long, how many square inches are in the total surface area (inside, outside, and ends) of the pipe? Leave answer in terms of π.

Geometry Test I Solutions

1. $\dfrac{(6^2-5^2)+(4^2-3^2)+(2^2-1^2)}{36}=\dfrac{7}{12}$ or assign a grid and notice that 21 of the 36 squares are shaded in.



2. $(\overline{PQ})^2=(\overline{RP})^2+(\overline{RQ})^2$

$169=25+(\overline{RQ})^2$

$\overline{RQ}=12$

Since $\overline{RQ}\parallel\overline{XY}$, $\overline{XY}=12$.

3. $\dfrac{2^3}{3^3}=\dfrac{200}{x}\Rightarrow 8x=200(27)$

$x=675$

4. On the "square", the four rounded portions have a combined perimeter of $2\pi r$, while the straight portions are each 4. Thus, the perimeter is $4\pi+16$.

On the "parallelogram", the perimeter is the same as that of the "square" for the rounded portions as well as the straight. Thus, perimeter is $4\pi+16$.

The difference, then, is zero.

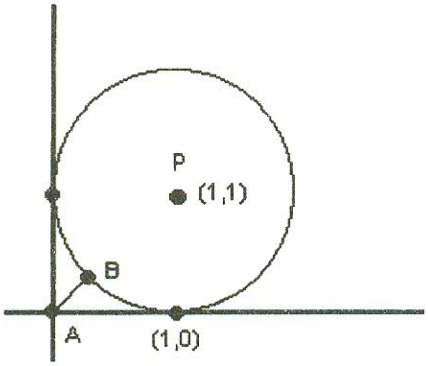
5. 4 semicircles would yield a circumference of 2 circles. Thus, $4\pi r=3\pi$, $r=\dfrac{3}{4}$. A side of the square would be $2\left(\dfrac{3}{4}\right)$ or $\dfrac{3}{2}$. $\overline{AD}$ is the diagonal of a square or the hypotenuse of a $45°-45°-90°$ triangle. $\overline{AD}=\dfrac{3}{2}\sqrt{2}$.

6. LMNO is a rhombus with each side $\dfrac{5}{2}\sqrt{5}$. The area of a rhombus $=\dfrac{1}{2}$ the product of the diagonals.

$LN=\sqrt{(5\sqrt{2})^2+5^2}=5\sqrt{3}$

$MO=$ diagonal of a face $=5\sqrt{2}$

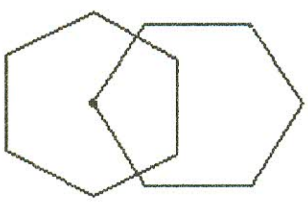
$Area=\dfrac{1}{2}(5\sqrt{2})(5\sqrt{3})=\dfrac{25}{2}\sqrt{6}$



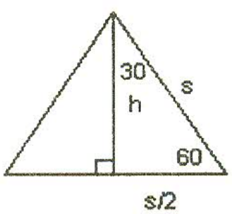
7. By Pythagorean theorem,

$\overline{AP} = \sqrt{2}$

$\overline{AB} = \sqrt{2} - 1$



8. If A_1 is the area of the first hexagon, A_2 is the area of the second, and A_{12} is the area of the overlapping region, the difference in areas is $(A_1 - A_{12}) - (A_2 - A_{12}) = A_1 - A_2$. Since the hexagons are congruent, $A_1 - A_2 = 0$.



9. First, find s.

$$h = \frac{s}{2}\sqrt{3}$$

$$Area = \frac{1}{2}(s)\left(\frac{s}{2}\sqrt{3}\right) = \frac{1}{4}s^2\sqrt{3}$$

$$25\sqrt{3} = \frac{1}{4}s^2\sqrt{3}$$

$s = 10$ the perimeter of the triangle is 30.

If r is the side of a hexagon, $6r = 30$, $r = 5$.

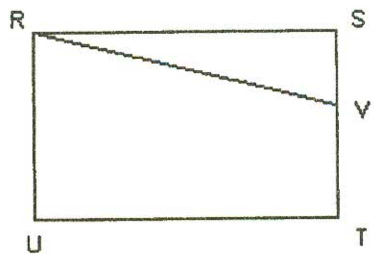
10. The inside surface area is $2\pi(2)(24)$ inches^2.

The outside is $2\pi(3)(24)$ inches^2.

Each end is $\pi(3)^2 - \pi(2)^2$ inches^2.

$Total = 96\pi + 144\pi + 10\pi = 250\pi$ inches^2.

West River Math Contest
Geometry Test II
2000

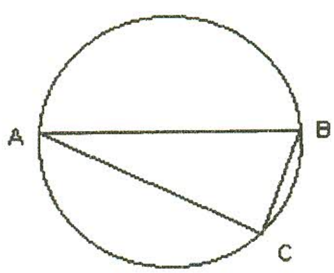


1. Line segment $\overline{RV}$ divides rectangle RSTU into two parts whose ratios are 8:1. Find the ratio of $\overline{VT} : \overline{SV}$

2. An oil company decides to increase the height of its cans of oil by 20% but to keep the volume the same. Find the ratio of the radius of the original oil can to the radius of the new one.

3. Find the length-to-width ratio of a restaurant menu (closed) if the rectangle formed by the open menu is similar to the rectangle formed by the closed menu.

4. If $\triangle ABC$ has sides of length 3, 4, and 5. Find the area of the circumscribed circle.



Geometry Test II Solutions

1. $\dfrac{AreaRSV}{AreaRSTU} = \dfrac{1}{8} = \dfrac{\frac{1}{2}\left(\overline{RS}\right)\left(\overline{SV}\right)}{\frac{1}{2}\left(\overline{UT}\right)\left(\overline{RU}+\overline{VT}\right)}$

But, $\overline{RS} = \overline{UT}$ and $\overline{RU} = \overline{SV} + \overline{VT}$.

$$\frac{1}{8} = \frac{\overline{SV}}{\overline{SV}+2\overline{VT}}$$

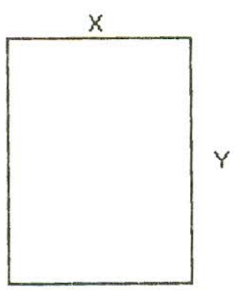
$8\overline{SV} = \overline{SV} + 2\overline{VT}$ and $7\overline{SV} = 2\overline{VT}$ $\quad \dfrac{\overline{VT}}{\overline{SV}} = \dfrac{7}{2}$

2. Let R be the radius of the old can. Let r be the new radius. Since volumes are the same,

$$\pi R^2 h = \pi r^2 (1.2)$$

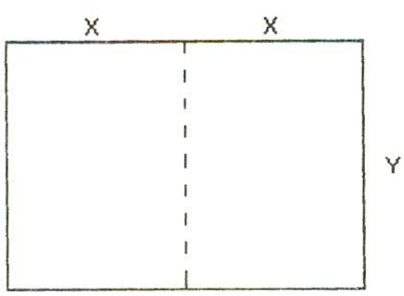
$R^2 = 1.2r^2 \qquad \dfrac{R}{r} = \dfrac{\sqrt{1.2}}{1} or \dfrac{\sqrt{6}}{\sqrt{5}} or \dfrac{\sqrt{30}}{5}$

3.



Closed Menu

Length = y
Width = x



Open Menu

Length = 2x
Width = y

Since the rectangles are similar: $\dfrac{y}{x} = \dfrac{2x}{y}$

$y^2 = 2x^2 \quad \dfrac{y^2}{x^2} = \dfrac{2}{1} \quad \dfrac{y}{x} = \dfrac{\sqrt{2}}{1}$

4. The triangle is a right triangle with its hypotenuse being a diameter of the circle.

$$Area = \pi r^2 = \pi\left(\frac{5}{2}\right)^2 = \frac{25\pi}{4}$$

West River Math Contest
Algebra II – Test I
2000

1. Solve for x, x a real number, in each of the following:

a. $|2x+7|<48$ b. $|3x-10|>27$ c. $|4x^2-2x|\le -x^2$

2. If $f(x)=\dfrac{x+2}{3x+4}$, determine $f^{-1}(x)$.

3. Solve for x: $\dfrac{x^2-4x+4}{2x^2+x-3}>0$.

4. What is the coefficient of $x^{1999}y^1$ in the expansion of $(2x-3y)^{2000}$? Terms of the form a^b are acceptable in your answer.

5. Give the nonzero polynomial of least degree having real coefficients, which has roots 3, -1+i.

6. Give the equation of the upward opening parabola through the points (x,y) = $(-1,6),(0,-1),(2,3)$.

7. Find all solution(s) x of $2\cdot 3^x+5=3^{1-x}$.

8. Completely factor the polynomial $x^4-2x^3-5x^2+6x$.

9. A chemical storeroom has an 80% alcohol solution and a 30% alcohol solution. How many milliliters of each should be used to obtain 50 milliliters of a 60% solution?

10. Determine the value(s) of k so that $x^4-3x^3+kx^2+19x-2k$ is divisible by (x-2).

11. In the diagram below, the ratio $\frac{s}{t}$ is said to be the "golden ratio" if the ratio of t to s is the same as the ratio of s to s+t. The golden ratio[1] crops up, among other places, in nature (e.g. in pinecones and sunflowers) and in art and architecture (especially Egyptian). Exactly determine the golden ratio $\frac{s}{t}$ (show your work).

12. The Body Mass Index (BMI) of a person provides an estimate of total body fat, which is linked to risk for certain diseases. BMI is defined as an individual's weight in kilograms (kg) divided by the square of their height in meters (m). BMI may also be computed as a constant K multiplied by weight in pounds (lbs) divided by the square of height in inches (in).

a. Write an expression for K (don't actually perform any multiplications or divisions) if

1 lb = .45359237 kg (exactly), and 1 in = .0254 m (exactly)

b. What are the units on K?

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[1]For more information see, for example, the Web site http://www.geocities.com/CapeCanaveral/Station/8228/.

Solutions - Algebra II - Test I

1. a. $-48 < 2x + 7 < 48 \Rightarrow -55 < 2x < 41 \Rightarrow -55/2 < x < 41/2$

   b. $3x - 10 > 27$ or $3x - 10 < -27$ so that $x > 37/3$ or $x < -17/3$

   c. The left-hand side of $|4x^2 - 2x| \le -x^2$ is bigger than or equal to zero while the right-hand side is less than or equal to zero. Consequently, solutions, if any exist, must correspond to x value(s) which make both sides of the inequality simultaneously zero. By inspection, x = 0 is the only solution.

2. $x = f(f^{-1}(x)) = \frac{f^{-1}(x)+2}{3f^{-1}(x)+4}$, giving $(3f^{-1}(x)+4)x = f^{-1}(x)+2$ or $f^{-1}(x)(3x-1) = 2-4x$ so $f^{-1}(x) = \frac{2-4x}{3x-1}$.

3. $0 < \frac{x^2-4x+4}{2x^2+x-3} = \frac{(x-2)^2}{(2x+3)(x-1)}$ which is positive for $x < -3/2, 1 < x < 2, x > 2$ (see below table; note that the rational expression is *identically* zero when x = 2).

Interval	Sign of Rational Expression
$(-\infty, -3/2)$	+
(-3/2, 1)	-
(1,2)	+
$(2, \infty)$	+

4. The expansion of $\underbrace{(2x-y)(2x-y)\cdots(2x-y)}_{2000\ terms}$ can be obtained by choosing a 2x or a –y from each of the 2000 terms taking the product and then summing over all possible such selections. There are 2000 ways of selecting 1999 2x values and 1 –y value, so the term involving $x^{1999}y$ is $2000(2x)^{1999}(-3y) = [-6000 \cdot 2^{1999}]x^{1999}y$ and the coefficient of $x^{1999}y$ is $-6000 \cdot 2^{1999}$. This problem can also be solved using the binomial expansion

$$(2x-y)^{2000} = \sum_{i=1}^{2000} \binom{2000}{i} (2x)^i (-y)^{2000-i}$$

and noting the desired term appears when i = 1999.

5. The desired polynomial is $(x-3)(x-(-1+i))(x-(-1-i)) = (x-3)(x^2+2x+2) = x^3 - x^2 - 4x - 6$ as the roots must be 3, $-1+i, -1-i$.

6. Substituting the three points into $y = ax^2 + bx + c$ we have

$$a - b + c = 6$$
$$c = -1$$
$$4a + 2b + c = 3$$

or

$$a - b = 7$$
$$c = -1$$
$$4a + 2b = 4$$

giving a = 3, b = -4, c = -1 and the desired parabola is $y = 3x^2 - 4x - 1$.

7. Multiplying through by 3^x we obtain $0 = 2\cdot 3^{2x} + 5\cdot 3^x - 3 = (2\cdot 3^x - 1)(3^x + 3)$ which implies that $(2\cdot 3^x - 1) = 0$ or $3^x = 1/2$, so that $x = \log_3(1/2) = -\log_3 2$. Equivalent answers include $-1/\log_2 3$, $-\log 2/\log 3$, and $-\ln 2/\ln 3$.

8. $x^4 - 2x^3 - 5x^2 + 6x = x(x^3 - 2x^2 - 5x + 6)$ and it is easy to see that 1 is a zero of the second polynomial in this last product. Dividing x-1 into this second polynomial, e.g. using synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -2 & -5 & 6 \\ & 1 & -1 & -6 & 0 \end{array},$$ we get $x^2 - x - 6$.

So $x^4 - 2x^3 - 5x^2 + 6x = x(x^3 - 2x^2 - 5x + 6) = x(x-1)(x^2 - x - 6) = x(x-1)(x-3)(x+2)$.

9. Letting x = amount of 80% alcohol solution (ml) and y = amount of 30% alcohol solution (ml), we have

Amount of alcohol (ml) = .8x + .3y = (50 ml)(60%) = 30 ml

Volume of solution (ml) = x + y = 50 ml

Solving these two equations in two unknowns we obtain x = 30 ml and y = 20 ml. So 30 ml of the 80% solution and 20 ml of the 30% solution are needed.

10. Upon dividing $x^4 - 3x^3 + kx^2 + 19x - 2k$ by (x-2) one obtains a quotient of $x^3 - x^2 + (k-2)x + (2k+15)$ with remainder 2k + 30 and 2k + 30 = 0 when k = -15.

11. $\frac{t}{s} = \frac{s}{s+t}$ implies $s^2 = ts + t^2$ or $s^2 - ts - t^2 = 0$ so $s = \frac{t \pm \sqrt{t^2 - (-4t^2)}}{2} = t\frac{1+\sqrt{5}}{2}$ or $\frac{s}{t} = \frac{1 \pm \sqrt{5}}{2}$. Since the ratio can't be negative, we have $\frac{s}{t} = \frac{1+\sqrt{5}}{2}$.

12. 4. $\text{BMI} = \left[\frac{\text{weight (kg)}}{(\text{height (m)})^2}\right] = K\left[\frac{\text{weight (lb)}}{(\text{height (in)})^2}\right]$, so

$$K = \left[\frac{\text{weight(kg)}}{\text{weight(lb)}}\right]\cdot\left[\left(\frac{\text{height(in)}}{\text{height(m)}}\right)^2\right] = \left[\frac{.45359237\text{ kg}}{1\text{ lb}}\cdot\left(\frac{1\text{ in}}{.0254\text{ m}}\right)^2\right] = \frac{.45359237}{.0254^2}\frac{\text{kg}\cdot\text{in}^2}{\text{lb}\cdot\text{m}^2}$$

West River Math Contest
Algebra II – Test II
2000

1. Evaluate ***one*** of the following:

a. $$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}$$

b. $$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$$

2. The digits in a three-digit number add to 12 and multiply to 42. If the ones and tens digits are reversed the resulting number is 36 higher than the original. What is the original number?

3. Determine the solution(s) to $\sqrt{2x+7}-\sqrt{x+4}-\sqrt{x-17}=0$.

4. A jar contains only cents, nickels, and dimes. The number of dimes is 2 more than the number of nickels, and the number of cents is 8 more than the number of dimes. How many of each denomination can be found in the jar, if the total value is $4.30?

5. If the sum of the digits of a number is divisible by three (without remainder), then this number is also divisible by three (without remainder). For example, 912 is divisible by three since 9 + 1 + 2 = 12 is divisible by three. Verify this property for any three digit number. That is, given a number abc (a is the hundreds unit, b the tens unit, and c the ones unit) show that it is divisible by 3 if a + b + c is divisible by 3.

Solutions - Algebra II - Test II

1. a. Let the expression be denoted by x. Note that $\frac{1}{x-1}=x$. Consequently $x^2-x-1=0$ and $x=\frac{1+\sqrt{5}}{2}$.

b. Let the expression be denoted by x. Note that $x^2-1=x$. Consequently $x^2-x-1=0$ and $x=\frac{1+\sqrt{5}}{2}$.

2. Denoting the number by xyz (i.e. the hundreds digit is x, the tens digit is y, the ones digit is z) we have the following 3 equations in 3 unknowns:

$$x + y + z = 12$$
$$xyz = 42$$
$$(10y + z) + 36 = 10z + y$$

The last equation simplifies to 9 y + 36 = 9 z or z = y + 4. Together with the first equation this gives x + y + (y + 4) = 12 or x = 8 – 2y. Using the second equation we obtain (8 – 2y) y (y + 4) = 42. The only integer solution among the values 0, 1, . . ., 9 is y = 3. Therefore x = 2, z = 7 and the number is 237. (Alternatively, the cubic equation in y reduces to $y^3-16y+21=0$ which has the root y = 3.)

Another approach: The fact that the digits multiply to 42 means that the number is either a permutation of the digits in 237 or 167. That is, only 237, 273, 327, 373, 723, 732 and 167, 176, 617, 671, 716, 761 are possibilities. The reversal of the tens and ones units increasing the value by 36 then eliminates all but 237 (the fact that the digits add to 12 not being needed).

3. Start, for example, with $\sqrt{2x+7}-\sqrt{x+4}=\sqrt{x-17}$ and square both sides to obtain, after some simplification $2\sqrt{2x+7}\sqrt{x+4}=2x+28$. Square both sides again and simplify to obtain $4x^2-52x-672=0$. The left-hand side of this expression is $4(x^2-13x-168)=4(x+8)(x-21)$ so that x = -8 and x = 21 are potential solutions. Only x = 21 satisfies the original equation.

4. Letting D = # Dimes, N = # Nickels, and C = # Cents we have D = N+2 and C = D+8 which implies C = N + 10. Substituting D = N+2 and C = N + 10 into C + 5N + 10D = 430 we find N = 25. Consequently, D = 27 and C = 35.

5. The value of the number abc is 100a + 10b + c and

$$\frac{100a+10b+c}{3}=\frac{a+b+c}{3}+\frac{99a+9b}{3}$$

or

$$\frac{100a+10b+c}{3}=\frac{a+b+c}{3}+(33a+3b)$$

So the left-hand side is an integer when (a + b + c)/3 is an integer. That is, abc is divisible by 3 without remainder if (and only if) a + b + c is divisible by 3 without remainder.

West River Math Contest
Advanced Math – Test I
2000

1. Find $\log_4 8$.

2. Find $\dfrac{\log_7 1000}{\log_7 100}$.

3. Find the solution set for $\dfrac{(x^2-4)(x+4)}{(x-7)x} \geq 0$.

4. Find the solution set for $\sqrt{(x-5)^2} > 0$.

5. Express $\dfrac{\sin x}{1-\cos x}$ as the sum of two trigonometric functions.

6. Given that $\sin 2t = 2\sin t \cos t$, express $\dfrac{1}{\sin^2 t} + \dfrac{1}{\cos^2 t}$ in simplified terms of $\sin 2t$.

7. Given that the decibel measure, *D*, of a sound of intensity *I* is given by $D = 10\log\dfrac{I}{I_0}$, where $I_0 = 10^{-12}$ watt per square meter, find the intensity of a sound which has a decibel level of 30. Note that the log in the formula is logarithm base 10.

8. What is the next year *y* so that $i^y = 1$, but *y* is **not** a leap year? Note that a year *y* is a leap if *y* is a multiple of 4 ***except*** that if *y* is a multiple of 100 then it has to be a multiple of 400 to be a leap year.

9. What is the inverse of the following matrix?

$$\begin{bmatrix} 7 & 2 \\ 17 & 5 \end{bmatrix}$$

10. If your sock drawer contains 6 brown socks, 10 black socks, and 12 white socks and your roommate is sleeping so you don't want to turn on the light, how many socks do you need to take so that you are sure of having a pair of the same color socks?

11. If P(A) is the power set of the set A, that is the set of all subsets of the set A, and A = {1, 2, 3, 4}, which of the following are elements of P(A)? Use the letters a, b, c, d, e in your answer. That is, if a and c are elements of P(A) and the others are not, then the correct answer is "a,c".

a. {1, 3}
b. { } [the empty set]
c. 3
d. {2, {4}}
e. {1, 2, 3, 4}

12. Given that the sum of the first n positive integers [that is $1 + 2 + \ldots + n$] is $\frac{n(n+1)}{2}$, what is the sum of the first n odd positive integers?

13. Solve for θ with $0 < \theta < 2\pi$: $2\sin^2\theta + \sin\theta = 1$

14. Express $\frac{2-5i}{3+2i}$ in the form $a + bi$ where a and b are real numbers.

15. Given that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, express $\binom{n}{k} + \binom{n}{k-1}$ in the form $\binom{a}{b}$ where a and b are in terms of n and k.

Advanced Math – Test I – Solutions
2000

1. $8 = 4*2 = 4\sqrt{4} = 4^{3/2}$, so $\log_4 8 = 3/2$

2. $\dfrac{\log_7 1000}{\log_7 100} = \dfrac{\log_7 10^3}{\log_7 10^2} = \dfrac{3\log_7 10}{2\log_7 10} = \dfrac{3}{2}$

3. The points at which this will change from true to false are the values that make either the top or bottom of the fraction zero. Thus, $x = -4, -2, 0, 2,$ and 7.

Test points in the intervals between these points and outside of these points:

$x = -1000$: Top is negative, bottom is positive, so false
$x = -3$: Top is positive, bottom is positive, so true
$x = -1$: Top is negative, bottom is positive, so false
$x = 1$: Top is negative, bottom is negative, so true
$x = 3$: Top is positive, bottom is negative, so false
$x = 1000$: Top is positive, bottom is positive, so true

This gives that the solution set is
$-4 \le x \le -2$ or $0 < x \le 2$ or $x > 7$.

4. $\sqrt{(x-5)^2} = |x-5|$, so this is true for all x except 5.

5.

$$\frac{\sin x}{1-\cos x} = \frac{\sin x}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x} = \frac{\sin x + \sin x\cos x}{1-\cos^2 x}$$
$$= \frac{\sin x + \sin x\cos x}{\sin^2 x} = \frac{1+\cos x}{\sin x} = \csc x + \cot x$$

6. $$\frac{1}{\sin^2 t} + \frac{1}{\cos^2 t} = \frac{\cos^2 t + \sin^2 t}{\sin^2 t\cos^2 t} = \frac{1}{\sin^2 t\cos^2 t} = \frac{1}{\left(\dfrac{\sin 2t}{2}\right)^2} = \frac{4}{\sin^2 2t}$$

7.

$$30 = 10\log\frac{I}{10^{-12}}$$

$$3 = \log I - \log 10^{-12} = \log I + 12$$

$$-9 = \log I$$

$$I = 10^{-9}$$

So, $I = 10^{-9}$ watt per square meter

8. Since $i^2 = -1$, $i^4 = 1$, so y is a multiple of 4. The next year that is a multiple of 4 but not a leap year is 2100.

9. The determinant of the given matrix is one, so the inverse is found by swapping the 5 and 7 and negating the 17 and 2. [Then, divide by the determinant, but dividing by 1 does not change anything.]

Thus, the inverse is $\begin{bmatrix} 5 & -2 \\ -17 & 7 \end{bmatrix}$.

10. It doesn't really matter how many of each color sock you have (as long as you have at least two), the important fact is the number of colors. Since there are three different colors, if you pick four socks you will get at least one pair the same color.

11. a, b, e

12. The nth odd positive integer is $2n - 1$, so the sum of the first n odd positive integers is $2\frac{n(n+1)}{2} - n = n^2 + n - n = n^2$.

Alternate solution: Consider 1 + 3 + 5 + 7 as (1 + 2 + 3 + 4) + (0 + 1 + 2 + 3). Generalizing this idea gives that the sum of the first n odd positive integers can be found by summing the first n positive integers and adding the sum of the first n – 1 positive integers. Thus, the sum is $\frac{n(n+1)}{2} + \frac{(n-1)n}{2} = \frac{n}{2}(n+1+n-1) = n^2$.

13.

$$2\sin^2\theta + \sin\theta = 1$$

$$2\sin^2\theta + \sin\theta - 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

$$\sin\theta = 1/2, -1$$

This gives that $\theta = \pi/6,\ 5\pi/6$, or $3\pi/2$

14. $$\frac{2-5i}{3+2i}=\frac{2-5i}{3+2i}\cdot\frac{3-2i}{3-2i}=\frac{-4-19i}{13}=\frac{-4}{13}+\frac{-19}{13}i$$

15.

$$\binom{n}{k}+\binom{n}{k-1}=\frac{n!}{k!(n-k)!}+\frac{n!}{(k-1)!(n-k+1)!}=\frac{n!}{(k-1)!(n-k)!}\left[\frac{1}{k}+\frac{1}{n-k+1}\right]$$

$$=\frac{n!}{(k-1)!(n-k)!}\left[\frac{n+1}{k(n-k+1)}\right]=\frac{(n+1)!}{k!(n+1-k)!}=\binom{n+1}{k}$$

West River Math Contest
Advanced Math – Test II
2000

1. Find the solution set for $\frac{2}{x+2} \le \frac{1}{x-5}$.

2. Find all zeros and all asymptotes (vertical and horizontal) of the following function. Clearly label which is which.

$$y = \frac{3x^2 - 5x + 2}{x^2 + 2x - 3}$$

3. Solve the following equation for x. Note that the bases on the logs are 4 and 2, they are not the same.

$$\log_4 x + \log_2 x = 15$$

4. How many ways are there for a photographer to arrange two candidates for president, their two vice presidential candidates and all four spouses in two rows of four people if both presidential candidates must be in the same row? We will assume that all four candidates are married even though that is not a constitutional requirement.

5. If you had a calculator that only had log base 10 and log base e buttons, how would you find $\log_2 1000$?

6. Express the two square roots of i in the form $a + bi$ where a and b are real numbers.

Advanced Math – Test II – Solutions
2000

1.

$$\frac{2}{x+2} \le \frac{1}{x-5}$$
$$\frac{2}{x+2} - \frac{1}{x-5} \le 0$$
$$\frac{2x-10-x-2}{(x+2)(x-5)} \le 0$$
$$\frac{x-12}{(x+2)(x-5)} \le 0$$

So, we look at the intervals determined by $x = -2, 5, 12$
Try points in each interval:

$x = -1000$: Top is negative, bottom is positive, so true
$x = 0$: Top is negative, bottom is negative, so false
$x = 10$: Top is negative, bottom is positive, so true
$x = 1000$: Top is positive, bottom is positive, so false

This gives that the solution set is $x < -2$ or $5 < x \le 12$.

2. $y = \dfrac{3x^2 - 5x + 2}{x^2 + 2x - 3} = \dfrac{(3x-2)(x-1)}{(x+3)(x-1)} = \dfrac{3x-2}{x+3}$ for $x \ne 1$.
Thus, the only zero is $x = 2/3$, the vertical asymptote is $x = -3$, and the horizontal asymptote is $y = 3$.

3. Since 4 is the square of 2, $4^y = 2^{2y}$ which means that the log base 4 of a number is half the log base 2 of that number. Thus, we have

$$\frac{1}{2}\log_2 x + \log_2 x = 15$$
$$\frac{3}{2}\log_2 x = 15$$
$$\log_2 x = 10$$
$$x = 2^{10} = 1024$$

4. There are 2 ways to pick a row for the candidates.
There are $4 \cdot 3 = 12$ ways to put two candidates in the row.
There are now 6 spots to fill with the remaining 6 people in any order, so $6! = 720$ ways to put in the rest of the people. Thus, the answer is $2 \cdot 12 \cdot 720 = 17280$.

5. $\log_2 1000 = \frac{\log 1000}{\log 2} = \frac{3}{\log 2}$ [or use ln similarly]

6. Solution #1: $i = (\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$, so cos 2θ = 0 and sin 2θ = 1. This means that 2θ = π / 2 or 5 π / 2. This gives or θ = π / 4 or 5 π / 4. So, the square roots of i are $\cos\theta + i\sin\theta = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ or $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

Solution #2: $i = (a+bi)^2 = a^2 - b^2 + 2abi$. This means that $a^2 - b^2 = 0$ and $2ab = 1$. Solving these simultaneously gives

$$b = \frac{1}{2a}$$

$$a^2 - \left(\frac{1}{2a}\right)^2 = 0$$

$$a^2 - \frac{1}{4a^2} = 0$$

$$4a^4 = 1$$

$$a^2 = 1/2$$

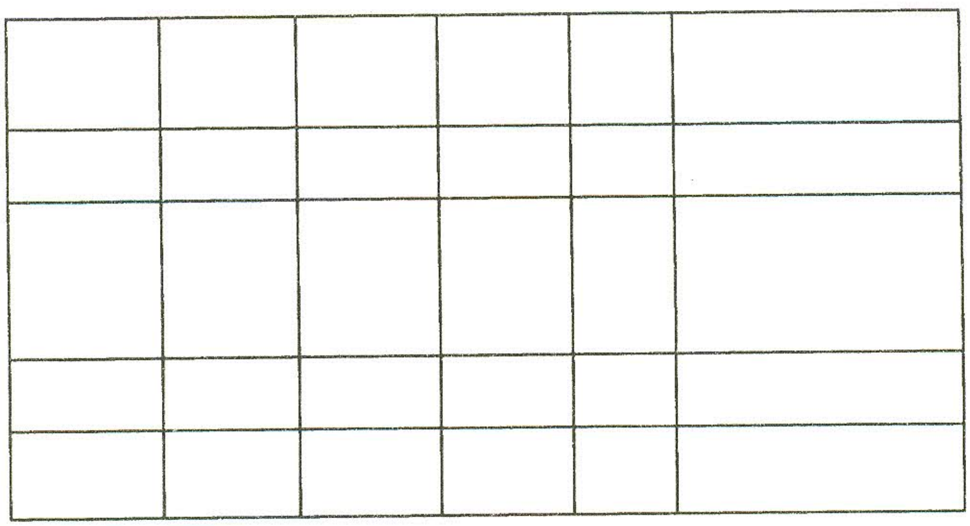
$$a = \pm\frac{\sqrt{2}}{2}$$

Thus, the two square roots of i are $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

West River Math Contest

Master's Exam

1. Find the number of rectangles formed by the following set of horizontal and vertical line segments pictured below. (Include squares as they too are rectangles.)



a. n people each toss a coin simultaneously, where n>= 3. If the coins fall one head and n-1 tails, the person with the head wins. If the coins fall one tail and n-1 heads, the person with the tail wins. If the coins fall in any other pattern then nobody wins and everyone tosses again. What is the probability that there is a winner the first time everyone tosses?

2.

a. n people each toss a coin simultaneously, where $n \geq 3$. If the coins fall one head and n-1 tails, the person with the head wins. If the coins fall one tail and n-1 heads, the person with the tail wins. If the coins fall in any other pattern then nobody wins and the toss is repeated. What is the probability that there is a winner the first time everyone tosses?

b. If five of us toss for winner, what is the probability that I win? Since I am as likely to win as any four of my opponents, the answer should be 1/5. Provide rigorous justification for this answer or show it to be incorrect.

3. Note that 3, 5, 7 are three consecutive odd numbers and all three are prime numbers. Are there any other sequences of three consecutive odd numbers all of which are prime? Find some examples of such if your answer is yes or justify your answer if your answer is no.

4. It is possible to find two numbers, a and b, both of which are rational, while a^b is irrational. For example, setting a=2 and b=1/2 and we see that $a^b=\sqrt{2}$ is an irrational number. Is it possible to find two numbers, a and b, both of which are irrational, while a^b is rational?

5. A two dimensional "cube" (that is, a square) has 4 edges. The vertices of the the "cube" can be labeled by the 2^2 ordered pairs of numbers: (0,0), (0,1), (1,1), (1,0).

Similarly, a three dimensional "cube" has 12 edges. The vertices of the "cube" can be labeled by the 2^3 ordered triples of numbers: (0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1).

An n-dimensional cube can be built by using the 2^n n- tuples of 0's and 1's.
How many edges does 6 dimensional cube have? Clearly describe your reasoning.

6. How many 10 digit numbers are there with the following properties:
only the digits 2 and 3 can be used and
no two adjacent digits can both be the digit 3.

Master's Exam Solutions

1. A rectangle needs two vertical sides and two horizontal sides.
The number of ways that one can select two vertical lines from the 7 above is "7 choose 2," that is, (7*6)/(2*1) = **21.**
Similarly, there are "6 choose 2" ways to select two horizontal lines from the 6 available; that is, (6*5)/(2*1) = **15.**
The product **21*15 = 315** gives the number of rectangles that can be built.

2. a. The probability that one of n people gets a head and all the others get a tail is $1/2^n$. Thus the probability that a toss results in one head and n-1 tails is $n * 1/2^n$.
Similarly, the probability that a toss results in one tail and n-1 heads is $n * 1/2^n$.
Thus the probability that either happens is $n * 1/2^n + n * 1/2^n = 2n/2^n$.

b. The probability that I win on the first toss is $2/2^n = 1/16$. If I don't win on the first toss, I still have a chance to win. I could win on the second toss if nobody wins on the first toss. The probability that nobody wins on the first toss is one minus the probability that somebody wins on the first toss, and by part a this is $1-2n/2^n = 1-10/32 = 11/16$. The probability that I win on the second toss is the probability that nobody wins on the first toss times the probability that I win on the second toss, which is 11/16 * 1/ 16.

If I don't win on either of the first two tosses, I could win by winning on the third toss. The probability that I win on the third toss is the probability that nobody wins the first toss times the probability that nobody wins the second toss times the probability that I win the third toss which is $(11/16)^2 * 1/16$.

In like manner, the probability that I win on exactly the kth toss is $(11/16)^{k-1} * 1/16$. Thus the likelihood that I win is the sum of the probabilities that I win each of the k tosses, for k = 1, 2, 3, ……

Thus, $\sum_{k=1}^{\infty}(11/16)^{k-1} * 1/16 \quad = \quad \frac{1}{1-\frac{11}{16}} * \frac{1}{16} \quad = \quad \frac{1}{5}$ is the requested probability.

3. This is not possible. Of any three consecutive odd numbers, one is always a multiple of 3. If the first of the three consecutive odd numbers is not a multiple of 3, then it leaves a remainder of either 1 or 2 when divided by three. If the remainder is 1, the second consecutive odd number, which is 2 greater than the first, is divisible by 3. If the remainder is 2, the third consecutive odd number, which is 4 greater than the first, is divisible by 3. Thus, one of the three consecutive odd numbers must be divisible by 3.

4. Yes. Let $x = \sqrt{2}^{\sqrt{2}}$. Is x rational or irrational? If x is rational, let a = b= $\sqrt{2}$ and we have an example showing that a^b is a rational number for a and b irrational. If x is irrational, let a = x and let b = $\sqrt{2}$. Then note $a^b = x^{\sqrt{2}} = 2$ and we have an example showing a^b is a rational number for a and b irrational.

5. An (n+1)-dimensional cube can be formed from an n-dimensional cube by taking two copies of the n-dimensional cube and forming edges between corresponding vertices. Thus, if g(n) is the number of edges in an n-dimensional cube, $g(n+1)= 2^n + 2*g(n)$.

Since g(3)= 12 (a three dimensional cube has 12 edges) we iterate: $g(4)= 2*12+2^3=32$; $g(5)=2*32+2^4=80$; $g(6)=2*80+2^5 = 192$

More formally, consider the following.

Two vertices are adjacent if all coordinates but one are identical. Edges are formed between adjacent vertices. To count the number of edges in a cube, we need to count the number of pairs of vertices that differ in exactly one coordinate.

If we look at a single n-tuple, there are n other n-tuples that differ from this one in exactly one coordinate - one for each coordinate position where the difference might occur. Since there are 2^n vertices, this gives $n*2^n$ ordered pairs. However, the order in which we look at the vertices that form an edge is irrelevant so we really want to look at unordered pairs. That is, we need to divide by two to get the true number of edges. Thus, the count of edges is given by $n*2^n/2 = n*2^{n-1}$. For n = 6, the count is $6*2^5 = 6*32 = 192$.

6. Let N(k) = the number of k-digit numbers that have the desired properties.
Let N(2,k) = the number of k-digit numbers that begin with 2 and have the desired properties.
N(2, k) = N(k-1) since the remaining k-1 digits can be any k-1 digit number with the desired properties.
Let N(3,k) = the number of k-digit numbers that begin with 3 and have the desired properties.
N(3,k) = N(k-2) since the second digit must be a 2 and the remaining k-2 digits can be any k-2 digit number with the desired properties.

Then N(k) = N(2,k) + N(3,k) = N(k-1) + N(k-2). Applying this when k=10, we have:

N(10) = N(9) + N(8)
= (N(8) + N(7)) + N(8) = 2*N(8) + N(7)
= 2*(N(7) + N(6)) + N(7) = 3*N(7) + 2*N(6)
= 3*(N(6) + N(5)) + 2*N(6) = 5*N(6) + 3*N(5)
= 5*(N(5) + N(4)) + 3*N(5) = 8*N(5) + 5*N(4)
= 8*(N(4) + N(3)) + 5*N(4) = 13*N(4) + 8*N(3)
= 13*(N(3) + N(2)) + 8*N(3) = 21*N(3) + 13*N(2)
= 21*(N(2) + N(1)) + 13*N(2) = 34*N(2) + 21*N(1)
= 34 *3 + 21*2 = 102 + 42 = 144

since N(2) = # 2 digit numbers (22, 23, 32) = 3 and N(1) = # 1 digit numbers (2, 3) = 2