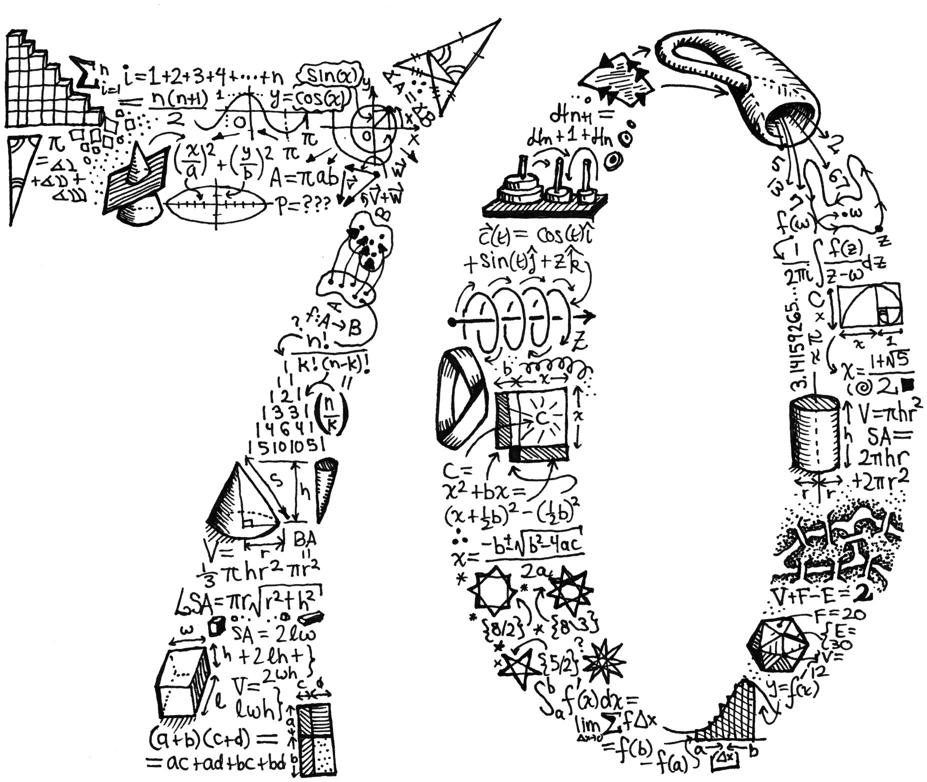
70th Annual

West River Math Contest

May 9, 2022

South Dakota Mines



This event is free for our competitors as funding is covered by the generous donations from our community and support across our South Dakota Mines campus. **Thank you sponsors!**

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70th West River Math Contest

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produced by

Department of Mathematics

2022 Test Authors include Professors

Debra Bienert
Karen Braman
Brent Deschamp
Patrick Fleming
Marti Garlick
Peter Grieve
Erin Handberg
Roger Johnson
Donald Teets

Thanks to our print editor Brent Deschamp and database support Jeffrey Winter. Additional thanks to all faculty volunteering their time for grading tests on the day of the contest, without whom this event would not take place!

Special thank you to our retired faculty, Julie Dahl and Debra Bienert, for running this contest over the years.

In addition, thank you to the teachers who encourage their students to compete and enrich a love of mathematics in the next generation! We look forward to continuing this event for your students for years to come, please contact our event coordinators, Michelle and Tristin, if you have any questions.

Michelle.Richard-Greer@sdsmt.edu Tristin.Lehmann@sdsmt.edu

Contents

2022 West River Math Contest
Algebra I – Exam I

Record the correct answer on the answer sheet.

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1. Find the mean of the set of numbers $\{2, 2, -2, -2, 12, -10\}$.

2. If one number is four times as large as another number and their sum is 55, what are the two numbers?

3. Solve the following system.

$$4x + 2y = 24$$
$$5x - 4y = 17$$

4. Use the discriminant to classify the number and type of roots for the quadratic $3x^2 + 7x + 4$. Report the value of the discriminant, as well as the classification. Do not compute the actual roots.

5. Find the vertex for the parabola given by $y = x^2 + 4x + 9$. Report your answer in the form (x, y).

6. Completely reduce the following fraction.

$$\frac{\frac{11x+66}{10x+7}}{\frac{x^2+11x+30}{10x+7}}$$

7. Consider the following modifications for a function: Given a function $f(x)$, we want to shift the graph of $f(x)$ to the left by 7, shift the graph of $f(x)$ up by 8, stretch the graph of $f(x)$ vertically by a factor of 6, and flip the graph of $f(x)$ vertically (reflect it about the x-axis). Modify $f(x)$ so that the new equation includes these modifications.

8. If

$$f(x) = \begin{cases} 4|x-10| + 3x & x < 5 \\ x + 5 & 5 \le x \le 7 \\ 4x^2 + 10 & x > 7 \end{cases},$$

find $f(9)$.

9. Write the expression

$$(3x - 2)(2 - 2x)(2x + 3)$$

as a polynomial in standard form.

10. Find the quotient and remainder for

$$(20x^4 + 8x^3 - 30x^2 + 8x + 11) \div (4x^2 - 2).$$

2022 West River Math Contest
Algebra I – Exam I
Solutions

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1. Find the mean of the set of numbers $\{2, 2, -2, -2, 12, -10\}$.

   Solution:
   $$\overline{x} = \frac{2+2-2-2+12-10}{6} = \frac{2}{6} = 0.33333.$$

2. If one number is four times as large as another number and their sum is 55, what are the two numbers?

   Solution: Let a and b be the two numbers. Then
   $$\begin{aligned} 4a &= b \\ a+b &= 55 \end{aligned}$$
   $$\begin{aligned} a+b &= 55 \\ a+4a &= 55 \\ 5a &= 55 \\ a &= 11 \end{aligned}$$
   $$\begin{aligned} b &= 4a \\ &= 4 \cdot 11 \\ &= 44. \end{aligned}$$
   Thus $a = 11$ and $b = 44$.

3. Solve the following system.
   $$\begin{aligned} 4x+2y &= 24 \\ 5x-4y &= 17 \end{aligned}$$

   Solution: Solve for x in equation 1.
   $$\begin{aligned} 4x+2y &= 24 \\ 4x &= 24-2y \\ x &= \frac{24-2y}{4}. \end{aligned}$$

Plug x into equation 2.

$$\begin{aligned} 5x - 4y &= 17 \\ 5\left(\frac{24 - 2y}{4}\right) - 4y &= 17 \\ 5(24 - 2y) - 16y &= 68 \\ 120 - 10y - 16y &= 68 \\ -26y &= 68 - 120 \\ -26y &= -52 \\ y &= 2 \end{aligned}$$

Plug y back in to find x.

$$\begin{aligned} x &= \frac{24 - 2(2)}{4} \\ x &= \frac{20}{4} \\ x &= 5. \end{aligned}$$

The solution is $x = 5$ and $y = 2$.

4. Use the discriminant to classify the number and type of roots for the quadratic $3x^2 + 7x + 4$. Report the value of the discriminant, as well as the classification. Do not compute the actual roots.

   Solution: The discriminant is

$$\begin{aligned} D &= 7^2 - 4(3)(4) \\ &= 49 - 48 \\ &= 1. \end{aligned}$$

   Since $D > 0$, then the quadratic has two real roots.

5. Find the vertex for the parabola given by $y = x^2 + 4x + 9$. Report your answer in the form (x, y).

   Solution:

$$\begin{aligned} x^2 + 4x + 9 &= (x^2 + 4x + (2)^2) - 4 + 9 \\ &= (x + 2)^2 + 5. \end{aligned}$$

   The vertex is at $(-2, 5)$.

6. Completely reduce the following fraction.

$$\frac{\dfrac{11x + 66}{10x + 7}}{\dfrac{x^2 + 11x + 30}{10x + 7}}$$

Solution:

$$\frac{\frac{11x+66}{10x+7}}{\frac{x^2+11x+30}{10x+7}} = \frac{11x+66}{10x+7}\cdot\frac{10x+7}{x^2+11x+30}$$
$$= \frac{11x+66}{x^2+11x+30}$$
$$= \frac{11(x+6)}{(x+5)(x+6)}$$
$$= \frac{11}{x+5}.$$

7. Consider the following modifications for a function: Given a function $f(x)$, we want to shift the graph of $f(x)$ to the left by 7, shift the graph of $f(x)$ up by 8, stretch the graph of $f(x)$ vertically by a factor of 6, and flip the graph of $f(x)$ vertically (reflect it about the x-axis). Modify $f(x)$ so that the new equation includes these modifications.

   Solution: Shifting a function to the left has the form $f(x+h)$, and so we need $f(x+7)$. Shifting a function up has the form $f(x)+h$, and so now we have $f(x+7)+8$. Stretching a function has the form $af(x)$, but we do not want to stretch the vertical shift, and so we have $6f(x+7)+8$. Finally, we want to reflect the function about the x-axis, which has the form $-f(x)$. We do not want to reflect the vertical shift, though, so we now have $-6f(x+7)+8$. Thus our final equation is
$$-6f(x+7)+8.$$

8. If
$$f(x) = \begin{cases} 4|x-10|+3x & x<5 \\ x+5 & 5\le x\le 7 \\ 4x^2+10 & x>7 \end{cases},$$
   find $f(9)$.

   Solution: Since $9>7$, then
$$\begin{aligned} f(9) &= 4(9)^2+10 \\ &= 4\cdot 81+10 \\ &= 324+10 \\ &= 334. \end{aligned}$$

9. Write the expression
$$(3x-2)(2-2x)(2x+3)$$
   as a polynomial in standard form.

   Solution:
$$\begin{aligned} (3x-2)(2-2x)(2x+3) &= (6x-6x^2-4+4x)(2x+3) \\ &= (-6x^2+10x-4)(2x+3) \\ &= -12x^3-18x^2+20x^2+30x-8x-12 \\ &= -12x^3+2x^2+22x-12. \end{aligned}$$

10. Find the quotient and remainder for

$$(20x^4 + 8x^3 - 30x^2 + 8x + 11) \div (4x^2 - 2).$$

Solution: Note that the subtraction in each step is implied.

$$
\begin{array}{r|rrrrr}
 & 5x^2 & +2x & -5 & & \\
\cline{2-6}
4x^2 - 2 & 20x^4 & +8x^3 & -30x^2 & +8x & +11 \\
\multicolumn{1}{r}{} & 20x^2 & & -10x^2 & & \\
\cline{2-6}
\multicolumn{1}{r}{} & & 8x^3 & -20x^2 & +8x & +11 \\
\multicolumn{1}{r}{} & & 8x^3 & & -4x & \\
\cline{3-6}
\multicolumn{1}{r}{} & & & -20x^2 & +12x & +11 \\
\multicolumn{1}{r}{} & & & -20x^2 & & +10 \\
\cline{4-6}
\multicolumn{1}{r}{} & & & & 12x & +1
\end{array}
$$

Thus the quotient is $5x^2 + 2x - 5$, and the remainder is $12x + 1$.

2022 West River Math Contest
Algebra I – Exam II

Record the correct answer on the answer sheet.

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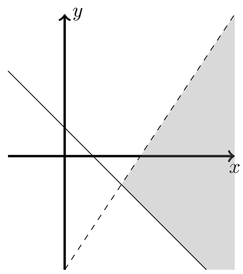
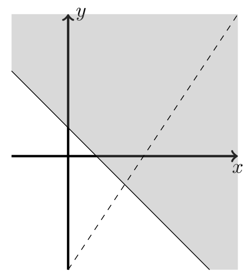
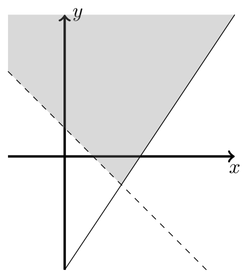
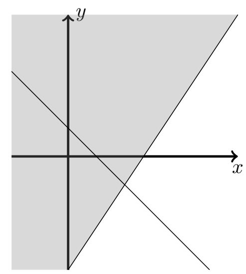
1. Plot the solution to the inequalities $y \geq -2x+3$ and $y < 2x-4$. Choose the graph that most closely matches your graph to report on the answer sheet.

(a) 

(b) 

(c) 

(d) 

2. Consider the sequence $3, 9, 27, \ldots$. Identify whether the sequence is arithmetic or geometric, and then state the next two terms in the sequence.

3. Completely simplify the following expression.

$$\frac{x^5 \cdot w^{1/3} \cdot \sqrt{(yz)^4} \cdot (y^{35}z^{25})^{1/5}}{(xz)^4 \cdot \sqrt[3]{wy^{21}}}$$

4. Solve for x in the equation $9y + 6xy = 24 - 6y + 11xy^2$.

2022 West River Math Contest
Algebra I – Exam II
Solutions

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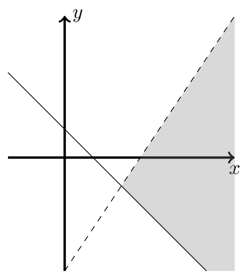
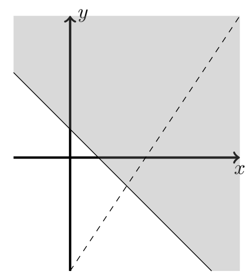
1. Plot the solution to the inequalities $y \geq -2x+3$ and $y < 2x-4$. Choose the graph that most closely matches your graph to report on the answer sheet.

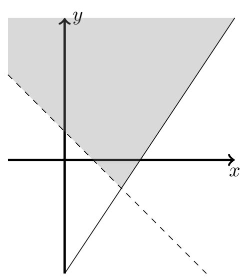
(a)



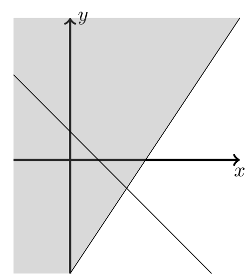
(b)



(c)



(d)



Solution: We will call $y \geq -2x+3$ equation (1) and $y < 2x-4$ equation (2). Since (1) has a negative slope and it allows for equality, we should see in the correct plot that the line with negative slope has a solid line. Since (2) has positive slope but does not allow for equality, that line should be dashed. This eliminates some of the possibilities.

Next, we should use a test point to determine which side of each line should be shaded. Let's use $(0,0)$. For (1), $0 \geq -2(0)+3$ is not true, and so the right side of (1) should be shaded, since that side of the line does not contain $(0,0)$. For (2), $0 < 2(0)-4$ is not true, and so the right side of (2) should be shaded. Since both inequalities need to be satisfied, we take the overlapping region common to both equations.

Thus plot (a) is the correct plot.

2. Consider the sequence $3, 9, 27, \ldots$. Identify whether the sequence is arithmetic or geometric, and then state the next two terms in the sequence.

Solution:

Since $9/3 = 3$ and $27/9 = 3$, then this sequence is geometric. The next two terms are thus $27 \cdot 3 = 81$, and $81 \cdot 3 = 243$.

3. Completely simplify the following expression.

$$\frac{x^5 \cdot w^{1/3} \cdot \sqrt{(yz)^4} \cdot (y^{35}z^{25})^{1/5}}{(xz)^4 \cdot \sqrt[3]{wy^{21}}}$$

Solution:

$$\begin{aligned}\frac{x^5 \cdot w^{1/3} \cdot \sqrt{(yz)^4} \cdot (y^{35}z^{25})^{1/5}}{(xz)^4 \cdot \sqrt[3]{wy^{21}}} &= \frac{x^5 \cdot w^{1/3} \cdot (yz)^2 \cdot (y^7 z^5)}{(xz)^4 \cdot \sqrt[3]{w} \cdot y^7} \\ &= \frac{x^5 \cdot w^{1/3} \cdot y^2 \cdot z^2 \cdot y^7 \cdot z^5}{x^4 \cdot z^4 \cdot w^{1/3} \cdot y^7} \\ &= \frac{w^{1/3}}{w^{1/3}} \cdot \frac{x^5}{x^4} \cdot \frac{y^2 \cdot y^7}{y^7} \cdot \frac{z^2 \cdot z^5}{z^4} \\ &= xy^2z^3.\end{aligned}$$

4. Solve for x in the equation $9y + 6xy = 24 - 6y + 11xy^2$.

Solution:

$$\begin{aligned}9y + 6xy &= 24 - 6y + 11xy^2 \\ 6xy - 11xy^2 &= 24 - 6y - 9y \\ x(6y - 11y^2) &= 24 - 15y \\ x &= \frac{24 - 15y}{6y - 11y^2}.\end{aligned}$$

2022 West River Math Contest
Algebra II – Exam I

Record the correct answer on the answer sheet.

---

1. Find an equation of the line that contains the point (1,2) and is perpendicular to the line $x + 3y = 6$. Write your answer in slope-intercept form.

2. Find the quotient and remainder for the following polynomial division: $\dfrac{x^4 - 3x^3 + 2x - 5}{x^2 - x + 1}$.

3. Simplify the following. Express your answer with positive exponents only. Assume the variables represent positive quantities.

$$\left(\frac{9x^2y^{1/3}}{x^{1/3}y}\right)^{1/2}$$

4. Find the real solutions of the equation: $\sqrt{2x+3} - \sqrt{x+2} = 2$.

5. The manager of Beans, Etc. decides to experiment with a new blend of coffee. She will mix some B grade Colombian coffee that sells for \$5 per pound with some A grade Arabica coffee that sells for \$10 per pound to get 100 pounds of new blend. The selling price of the new blend will be \$7 per pound. How many pounds of the B grade Colombian and how many pounds of the A grade Arabica coffees are required?

6. Suppose $f(x) = x^2 - 4x + c$ and $g(x) = \dfrac{f(x)}{3} - 4$. Find $f(3)$ if $g(-2) = 5$.

7. The function
$$f(x) = \frac{2x+1}{x-1}$$
is one-to-one. Find its inverse function.

8. Solve for x in $\sqrt[4]{\sqrt[3]{\sqrt{x-7} - 10} + 18} = 2$.

9. Solve the equation for x: $8^{-x+11} = 16^{2x}$.

10. Find k, if $(3x - 2k)(4x + 3k) = 12x^2 + kx - 96$.

11. Find the quotient and write the answer in scientific notation: $\dfrac{1.62 \times 10^{-4}}{4.5 \times 10^{-10}}$.

12. Solve for x: $\dfrac{3}{4}x - \dfrac{1}{5}\left(\dfrac{1}{2} - 3x\right) + 1 = \dfrac{1}{4}\left(\dfrac{1}{5}x + 6\right) - \dfrac{4}{5}$.

2022 West River Math Contest
Algebra II – Exam I
Solutions

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1. Find an equation of the line that contains the point (1,2) and is perpendicular to the line $x + 3y = 6$. Write your answer in slope-intercept form.

   Solution: First determine the slope of $x + 3y = 6$ by rewriting the equation of the line in slope-intercept form.

$$\begin{aligned} x + 3y &= 6 \\ 3y &= -x + 6 \\ y &= -\frac{1}{3}x + 2. \end{aligned}$$

   The slope of the given line is $-\frac{1}{3}$ and the slope of a perpendicular line is the negative reciprocal, or equal to 3. An equation of a line with slope 3 that contains the point (1,2) can be found by using the point-slope form with $m = 3, x_1 = 1$ and $y_1 = 2$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 3(x - 1) \\ y - 2 &= 3x - 3 \\ y &= 3x - 1. \end{aligned}$$

2. Find the quotient and remainder for the following polynomial division: $\dfrac{x^4 - 3x^3 + 2x - 5}{x^2 - x + 1}$.

   Solution: Divide using polynomial division. Note that the subtraction has been included in each step.

$$\begin{array}{r} x^2 - 2x - 3 \\ x^2 - x + 1 \overline{\smash{)}\; x^4 - 3x^3 + 0x^2 + 2x - 5} \\ \underline{-x^4 + x^3 - x^2 } \\ -2x^3 - x^2 + 2x \\ \underline{2x^3 - 2x^2 + 2x } \\ -3x^2 + 4x - 5 \\ \underline{3x^2 - 3x + 3} \\ x - 2 \end{array}$$

   The quotient is $x^2 - 2x - 3$, and the remainder is $x - 2$.

3. Simplify the following. Express your answer with positive exponents only. Assume the variables represent positive quantities.

$$\left(\frac{9x^2y^{1/3}}{x^{1/3}y} \right)^{1/2}$$

Solution: Recall that $(a^m)^n = a^{mn}$ and $(a \cdot b)^p = a^p \cdot b^p$. Thus

$$\left(\frac{9x^2y^{1/3}}{x^{1/3}y}\right)^{1/2} = \frac{9^{1/2}\left(x^2\right)^{1/2}\left(y^{1/3}\right)^{1/2}}{(x^{1/3})^{1/2}(y)^{1/2}} = \frac{3xy^{1/6}}{x^{1/6}y^{1/2}} = \frac{3x^{5/6}}{y^{1/3}}.$$

4. Find the real solutions of the equation: $\sqrt{2x+3} - \sqrt{x+2} = 2$.

   Solution:

$$\begin{aligned}
\sqrt{2x+3} - \sqrt{x+2} &= 2 \\
\sqrt{2x+3} &= \sqrt{x+2} + 2 \\
2x+3 &= x+2+4\sqrt{x+2}+4 \\
x-3 &= 4\sqrt{x+2} \\
x^2-6x+9 &= 16x+32 \\
x^2-22x-23 &= 0 \\
(x-23)(x+1) &= 0. \\
x = 23, &\quad x = -1.
\end{aligned}$$

   Because this is a radical equation, it may generate extraneous solutions, so both solutions must be checked.

   At $x = 23$; $\quad \sqrt{2x+3} - \sqrt{x+2} = 2, \quad \sqrt{46+3} - \sqrt{25} = 2, \quad 7-5 = 2, \quad 2 = 2.$

   At $x = -1$; $\quad \sqrt{2x+3} - \sqrt{x+2} = 2, \quad \sqrt{-2+3} - \sqrt{1} = 2, \quad 1-1 = 2, \quad 0 = 2.$

   The solution $x = 23$ checks but the solution $x = -1$ does not and must be discarded. The solution set is $\{23\}$ or $x = 23$.

5. The manager of Beans, Etc. decides to experiment with a new blend of coffee. She will mix some B grade Colombian coffee that sells for $5 per pound with some A grade Arabica coffee that sells for $10 per pound to get 100 pounds of new blend. The selling price of the new blend will be $7 per pound. How many pounds of the B grade Colombian and how many pounds of the A grade Arabica coffees are required?

   Solution: Let $a =$pounds of Grade A coffee

   Let $b =$pounds of Grade B coffee

   100 pounds of coffee is desired so we are able to write the equation $a + b = 100$ and solving for b, gives $b = 100 - a$. This provides a means to write an equation relating the revenue in terms of just the variable a.

$$\begin{aligned}
10a + 5(100-a) &= 7(100) \\
10a + 500 - 5a &= 700 \\
5a &= 200 \\
a &= 40.
\end{aligned}$$

   Substituting back in the first equation gives $b = 100 - 40 = 60$. Therefore, 40 lbs. of Grade A should be mixed with 60 lbs. of Grade B to get the desired blend.

6. Suppose $f(x) = x^2 - 4x + c$ and $g(x) = \dfrac{f(x)}{3} - 4$. Find $f(3)$ if $g(-2) = 5$.

Solution: The function $g(x)$ is written in terms of the function $f(x)$. Rewriting $g(x)$ by substituting $f(x)$ gives:

$$g(x) = \frac{x^2 - 4x + c}{3} - 4$$

Given $g(-2) = 5$ and substituting produces:

$$\begin{aligned}
\frac{(-2)^2 - 4(-2) + c}{3} - 4 &= 5 \\
\frac{4 + 8 + c}{3} &= 9 \\
12 + c &= 27 \\
c &= 15.
\end{aligned}$$

Knowing $c = 15$, solving for $f(3)$ gives the result:

$$f(3) = 3^2 - 4(3) + 15 = 9 - 12 + 15 = 12.$$

7. The function

$$f(x) = \frac{2x + 1}{x - 1}$$

is one-to-one. Find its inverse function.

Solution: The first step is to replace $f(x)$ with y and interchange the variables x and y.

$$x = \frac{2y + 1}{y - 1}$$

To find the inverse, solve for y.

$$\begin{aligned}
x(y - 1) &= 2y + 1 \\
xy - x &= 2y + 1 \\
xy - 2y &= x + 1 \\
y(x - 2) &= x + 1 \\
y &= \frac{x + 1}{x - 2}.
\end{aligned}$$

8. Solve for x in $\sqrt[4]{\sqrt[3]{\sqrt{x - 7} - 10} + 18} = 2$.

Solution: To solve, we need to isolate and then clear each radical. See below.

$$\begin{aligned}
\sqrt[4]{\sqrt[3]{\sqrt{x-7}-10}+18} &= 2 \\
\sqrt[3]{\sqrt{x-7}-10}+18 &= 16 \\
\sqrt[3]{\sqrt{x-7}-10} &= -2 \\
\sqrt{x-7}-10 &= -8 \\
\sqrt{x-7} &= 2 \\
x-7 &= 4 \\
x &= 11.
\end{aligned}$$

9. Solve the equation for x: $8^{-x+11} = 16^{2x}$.

   Solution: Rewriting both sides of the equation in terms of base 2 produces an equation that can be solved by equating the exponents of both sides.

$$\begin{aligned}
8^{-x+11} &= 16^{2x} \\
2^{3(-x+11)} &= 2^{4(2x)} \\
-3x+33 &= 8x \\
11x &= 33 \\
x &= 3.
\end{aligned}$$

10. Find k, if $(3x-2k)(4x+3k) = 12x^2 + kx - 96$.

    Solution: Expanding the left side and simplifying will give the information required to solve for k.

$$\begin{aligned}
(3x-2k)(4x+3k) &= 12x^2 + kx - 96 \\
12x^2 - 8xk + 9xk - 6k^2 &= 12x^2 + kx - 96 \\
xk - 6k^2 &= kx - 96 \\
-6k^2 &= -96 \\
k^2 &= 16 \\
k &= \pm 4.
\end{aligned}$$

11. Find the quotient and write the answer in scientific notation: $\dfrac{1.62 \times 10^{-4}}{4.5 \times 10^{-10}}$.

    Solution: First divide 1.62 by 4.5, as shown:

$$\begin{array}{r}
.36 \\
45\overline{)16.2} \\
\underline{135} \\
270 \\
\underline{270} \\
0
\end{array}$$

We now move our attention to the base 10 portion of the quotient. Dividing like bases requires we subtract the exponent in the denominator from the numerator as such:

$$10^{(-4-(-10))} = 10^6.$$

Our answer can now be written as 0.36×10^6. However this is not scientific notation so we need to move the decimal point one place to the right giving a final answer as 3.6×10^5.

12. Solve for x: $\frac{3}{4}x - \frac{1}{5}\left(\frac{1}{2} - 3x\right) + 1 = \frac{1}{4}\left(\frac{1}{5}x + 6\right) - \frac{4}{5}$.

Solution: Simplify and combine like terms to solve for x.

$$\frac{3}{4}x - \frac{1}{5}\left(\frac{1}{2} - 3x\right) + 1 = \frac{1}{4}\left(\frac{1}{5}x + 6\right) - \frac{4}{5}$$

$$\frac{3}{4}x - \frac{1}{10} + \frac{3}{5}x + 1 = \frac{1}{20}x + \frac{3}{2} - \frac{4}{5}$$

$$\frac{15x + 12x - x}{20} = \frac{1 - 10 + 15 - 8}{10}$$

$$\frac{26x}{20} = \frac{-2}{10}$$

$$26x = -4$$

$$x = -\frac{4}{26}$$

$$x = -\frac{2}{13}.$$

2022 West River Math Contest
Algebra II – Exam II

Record the correct answer on the answer sheet.

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1. Find the center and the radius of the circle: $2x^2 + 2y^2 - 12x + 8y - 24 = 0$.
2. Solve the equation: $\dfrac{2}{y+3} + \dfrac{3}{y-4} = \dfrac{5}{y+6}$.
3. Solve the equation for x: $\log_a(x) + \log_a(x-2) = \log_a(x+4)$.
4. Solve for x: $12x^{7/5} + 3x^{2/5} = 13x^{9/10}$.
5. Judy and Tom agree to share the cost of an \$18 pizza based on how much each ate. If Tom ate $\frac{2}{3}$ the amount that Judy ate, how much should each pay?
6. A piecewise function is defined as:

$$f(x) = \begin{cases} -2x+1 & \text{if} \quad -3 \le x < 1 \\ 2 & \text{if} \quad x = 1 \\ x^2 & \text{if} \quad x > 1 \end{cases}$$

   (a) Find $f(-2), f(1), f(2)$.
   (b) Find the domain of $f(x)$. Write your answer in interval notation.
   (c) Locate any y-intercepts.

2022 West River Math Contest
Algebra II – Exam II
Solutions

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1. Find the center and the radius of the circle: $2x^2 + 2y^2 - 12x + 8y - 24 = 0$.

   Solution: The standard form of the equation of a circle is written as $(x-h)^2+(y-k)^2 = r^2$, where (h, k) is the center and r is the radius. Therefore, completing the square is required for both the x components and the y components. To begin, move the constant to the right side and rearrange the terms on the left so the x and y variables are together.

   $$2x^2 - 12x + 2y^2 + 8y = 24.$$

   Because the coefficient of the squared terms is not 1, divide both sides by 2.

   $$x^2 - 6x + y^2 + 4y = 12.$$

   To create a perfect square for the x terms, divide the x coefficient by 2, square it, and add this to both sides. The same should be done for the y terms.

   $$x^2 - 6x + 9 + y^2 + 4y + 4 = 12 + 9 + 4.$$

   Rewriting the left side as two perfect squares tells us the center and the radius of the circle.

   $$(x - 3)^2 + (y + 2)^2 = 25.$$

   The center of the circle is $(3, -2)$, and the radius is 5.

2. Solve the equation: $\dfrac{2}{y+3} + \dfrac{3}{y-4} = \dfrac{5}{y+6}$.

   Solution: You first want to clear the fractions by multiplying the entire equation by the common denominator (least common multiple, LCM). This LCM is $(y + 3)(y - 4)(y + 6)$. Then it is a matter of combining like terms and solving for the variable y.

   $$\begin{aligned}
   \frac{2}{y+3} + \frac{3}{y-4} &= \frac{5}{y+6} \\
   (2)(y-4)(y+6) + 3(y+3)(y+6) &= 5(y+3)(y-4) \\
   2y^2 + 4y - 48 + 3y^2 + 27y + 54 &= 5y^2 - 5y - 60 \\
   36y &= -66 \\
   y &= \frac{-66}{36} \text{ or } \frac{-11}{6}.
   \end{aligned}$$

3. Solve the equation for x: $\log_a(x) + \log_a(x - 2) = \log_a(x + 4)$.

   Solution: The log terms on the left side of the equation can be combined recognizing that the sum of logs with the same base can be combined as a product of the arguments. Once this is done, both sides of the equation have logarithms with the same base and, therefore, their arguments must equate.

$$\begin{aligned}\log_a(x) + \log_a(x-2) &= \log_a(x+4)\\ \log_a((x)(x-2)) &= \log_a(x+4)\\ x^2 - 2x &= x + 4\\ x^2 - 3x - 4 &= 0\\ (x-4)(x+1) &= 0\\ x = 4 \quad x &= -1.\end{aligned}$$

However, the argument of a logarithm cannot be negative so the solution $x = -1$ must be discarded. The answer is simply $x = 4$.

4. Solve for x: $12x^{7/5} + 3x^{2/5} = 13x^{9/10}$.

   Solution:

$$\begin{aligned}12x^{7/5} + 3x^{2/5} &= 13x^{9/10}\\ 12x^{14/10} + 3x^{4/10} &= 13x^{9/10}\\ x^{4/10}(12x - 13x^{1/2} + 3) &= 0.\end{aligned}$$

   At this point we can conclude that one of the solutions is $x = 0$ based on the $x^{4/10}$ factor. For the portion in parentheses, we will use the substitution $u = x^{1/2}$ and rewrite as follows:

$$\begin{aligned}12u^2 - 13u + 3 &= 0\\ (4u-3)(3u-1) &= 0\\ u = \frac{3}{4}, \qquad u &= \frac{1}{3}.\end{aligned}$$

   Substituting $x^{1/2}$ back in for u and solving for x, we see that $x = u^2$. Therefore, $x = \left(\frac{3}{4}\right)^2$, or $x = \frac{9}{16}$, and $x = \left(\frac{1}{3}\right)^2$ or $x = \frac{1}{9}$. The complete solution set is $\{0,\ 9/16,\ 1/9\}$.

5. Judy and Tom agree to share the cost of an \$18 pizza based on how much each ate. If Tom ate $\frac{2}{3}$ the amount that Judy ate, how much should each pay?

   Solution: Let y = amount of pizza Judy ate, and let $\frac{2}{3}y$ = amount of pizza Tom ate.

   Judy and Tom ate an entire pizza which means we can solve for y.

$$\begin{aligned}y + \frac{2}{3}y &= 1\\ \frac{5}{3}y &= 1\\ y &= \frac{3}{5}.\end{aligned}$$

   Judy's cost will be $\frac{3}{5}$ of \$18, or $(18)\left(\frac{3}{5}\right) = \frac{54}{5} = 10\frac{4}{5} = \$10.80$.

   Tom's cost will be $\$18 - \$10.80 = \$7.20$.

6. A piecewise function is defined as:

$$f(x) = \begin{cases} -2x+1 & \text{if} \quad -3 \le x < 1 \\ 2 & \text{if} \quad x = 1 \\ x^2 & \text{if} \quad x > 1 \end{cases}$$

(a) Find $f(-2), f(1), f(2)$.

(b) Find the domain of $f(x)$. Write your answer in interval notation.

(c) Locate any y-intercepts.

Solution:

(a) To find $f(-2)$, observe that $-3 \le -2 < 1$, so when $x = -2$, the equation for f is given by $f(x) = -2x + 1$. Then

$$f(-2) = -2(-2) + 1 = 5.$$

Similarly, when $x = 1$, the equation for f is $f(x) = 2$. So,

$$f(1) = 2.$$

Finally, when $x = 2$, the equation for f is given by $f(x) = x^2$. Then

$$f(2) = 2^2 = 4.$$

(b) The domain of f is the union of the domains of each equation in the piecewise-defined function. So the domain of f is $[-3, 1) \cup \{1\} \cup (1, \infty) = [-3, \infty)$. The interval is $[-3, \infty)$ in interval notation, or $\{x : x \ge -3\}$ in set notation.

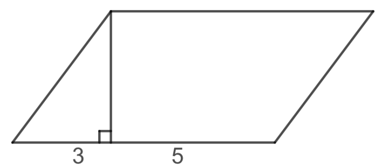
(c) The y-intercept of the graph of the function is at $f(0)$, or when $x = 0$. Because the equation for f when $x = 0$ is $f(x) = -2x + 1$, the y-intercept is $f(0) = -2 \cdot 0 + 1 = 1$. The y-intercept is (0,1).

2022 West River Math Contest
Geometry – Exam I

Record the correct answer on the answer sheet.

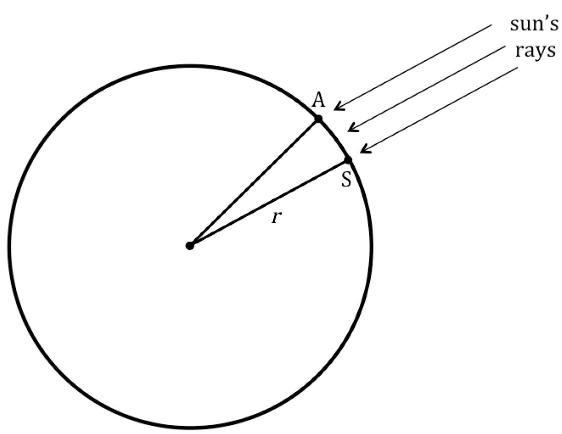
1. A circle of radius 6 has a central angle of 60°. Find the area of the sector this central angle cuts from the circle.

2. The parallelogram shown below has a perimeter of 26. Find its area.

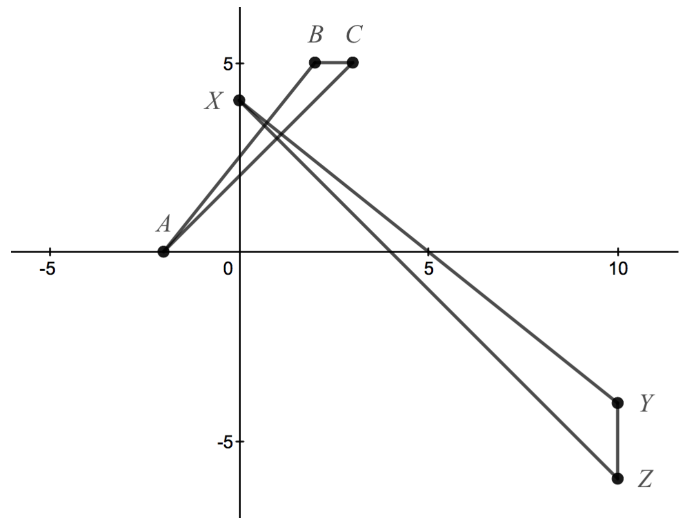


3. A regular octahedron has 8 faces that are equilateral triangles. How many edges and vertices does it have?

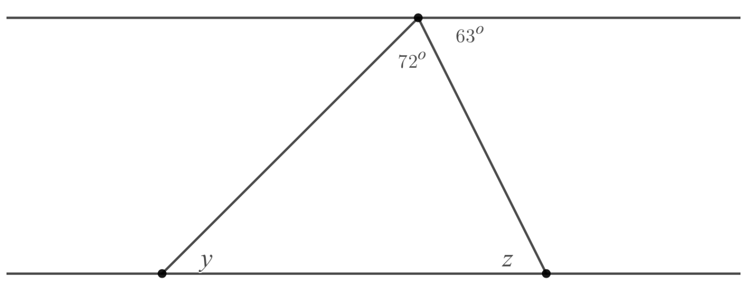
4. On the summer solstice, Eratosthenes observed that the sun's rays shown vertically and directly into a well at Syrene. In Alexandria, at the same date and time, the sun cast a shadow, and Eratosthenes calculated that the sun's rays fell at an angle of 7.2° from vertical. If Alexandria and Syrene are 490 miles apart, use this information to calculate the radius of the Earth.



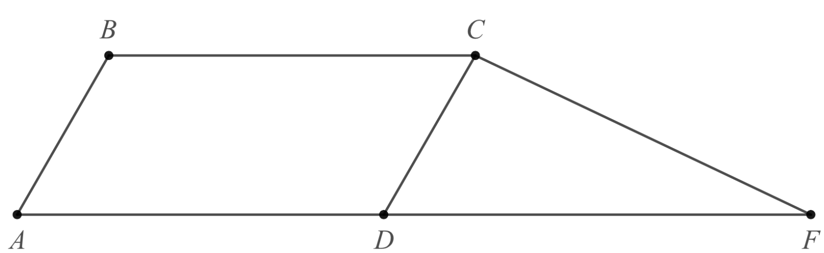
5. Let $\triangle ABC$ have vertices at $A = (-2, 0)$, $B = (2, 5)$, and $C = (3, 5)$, and let $\triangle XYZ$ have vertices at $X = (0, 4)$, and $Y = (10, -4)$, and $Z = (10, -6)$. Are these triangles similar? If so, what is the scaling factor?



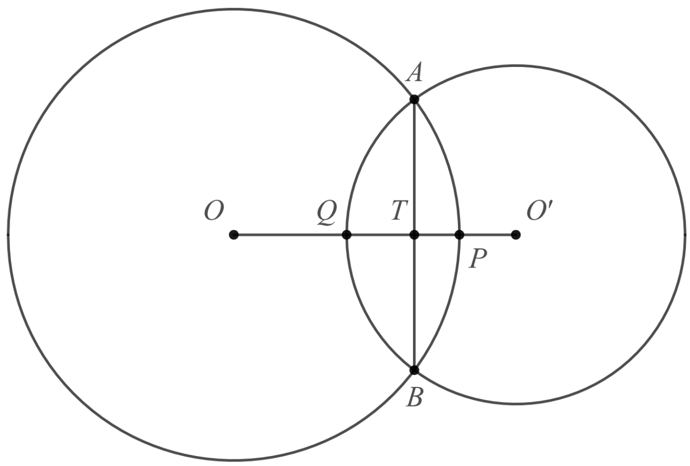
6. The vertices of a triangle lie on two parallel lines as shown in the figure below. Given the values of 63° and 72°, determine the angles y and z.



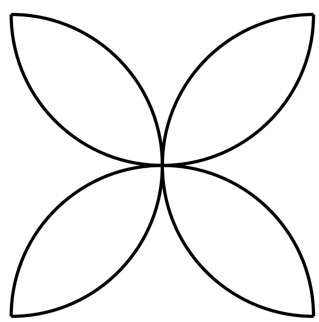
7. Assume that $ABCD$ in the following diagram is a parallelogram and that the points A, D, and F are collinear. If $\angle B = 120°$ and $\angle F = 25°$, find $\angle DCF$.



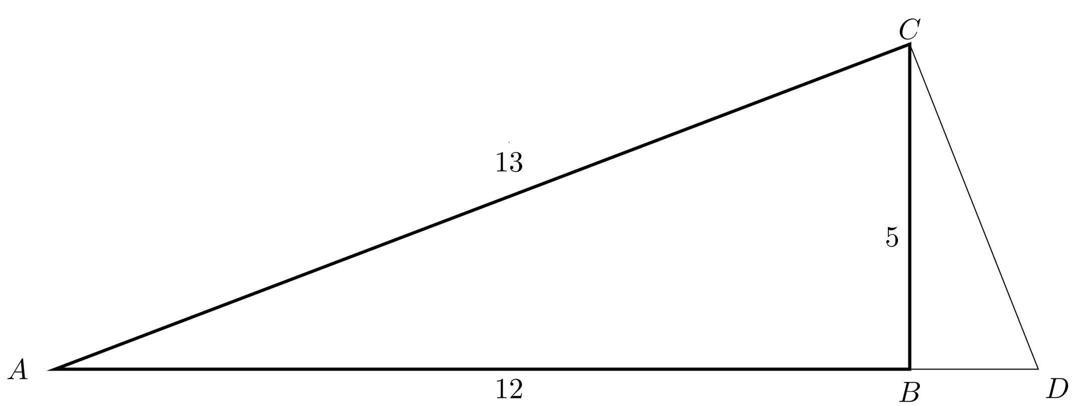
8. Two circles with centers of O and O' intersect at A and B. The length of the common chord $\overline{AB} = 10$. The line joining the centers intersects the circles at P and Q. If $\overline{PQ} = 2$ and the radius of one circle is 13, find the radius of the other circle. (Figure not drawn to scale.)



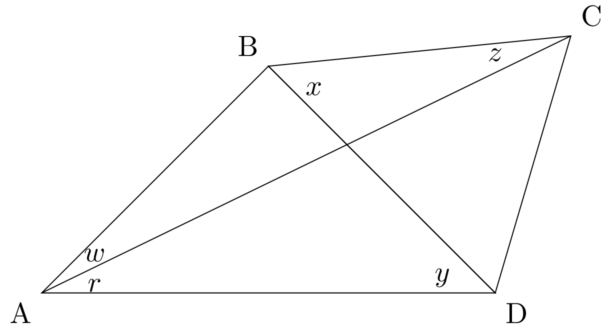
9. The intersction of four semicircles of radius r defines a flower shape with four petals. What is the radius r if the total area of the flower is 2.



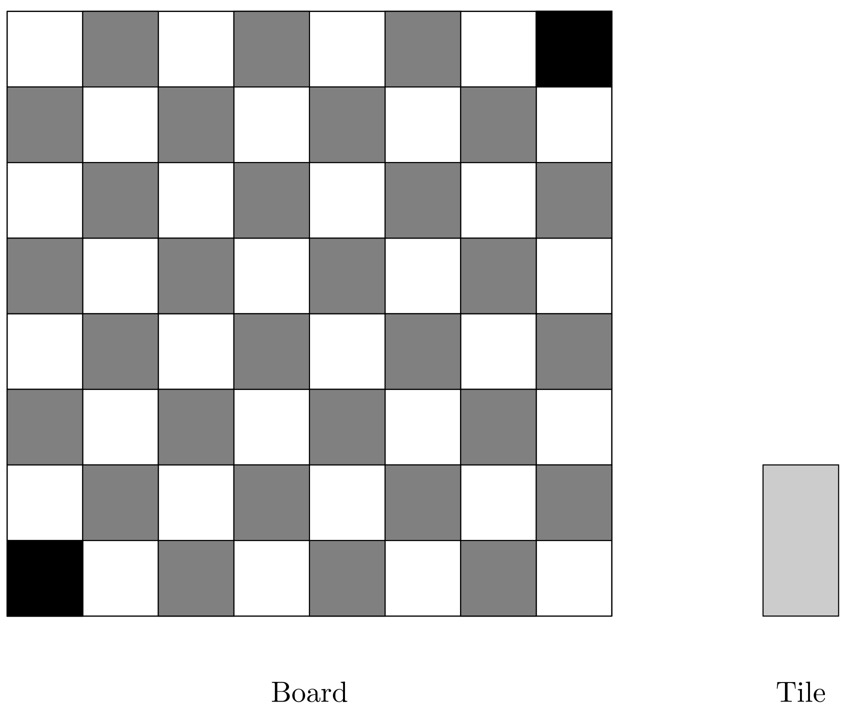
10. A triangle ABC with side lengths of 12, 5, and 13 is extended to a similar triangle ACD as shown below. What is the area of the larger triangle ACD?



11. Trapezoid ABCD has the diagonals connecting the corners and the resulting angles are: $w = 30°$, $r = 35°$, $y = 45°$, and $z = 25°$, where positions of r, w, x, y and z are shown in the figure below. Find the value of the angle labeled x.



12. How many 2×1 tiles can be placed on an 8×8 "checker board" after two of the opposite corners of the board have been removed?



2022 West River Math Contest
Geometry – Exam I
Solutions

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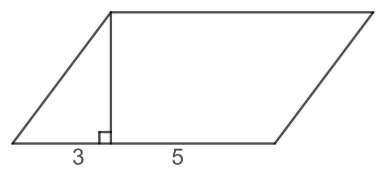
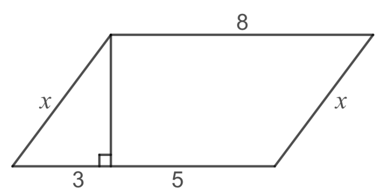
1. A circle of radius 6 has a central angle of 60°. Find the area of the sector this central angle cuts from the circle.

   Solution: The area of a sector is $A = \frac{1}{2}r^2\theta$, where θ is the area of the sector in radians. Thus, $60° = 60°\left(\frac{\pi}{180°}\right) = \frac{\pi}{3}$ radians, and $A = \frac{1}{2}(6)^2\left(\frac{\pi}{3}\right) = 6\pi$.

2. The parallelogram shown below has a perimeter of 26. Find its area.



   Solution: Pairs of parallel sides must have the same length. Thus, the perimeter can be described by $P = 16 + 2x = 26$, or $x = 5$.

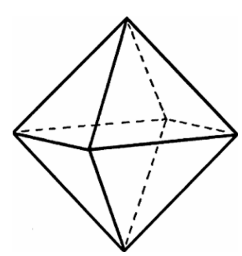


   By the Pythagorean theorem, the right triangle shown has a height of 4. Thus, the parallelogram has an area of $A = bh = (8)(4) = 32$.

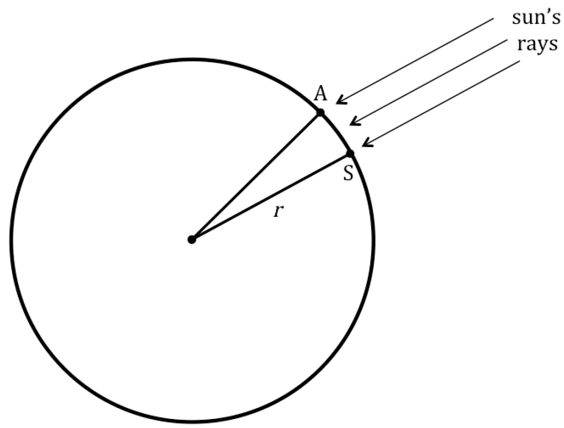
3. A regular octahedron has 8 faces that are equilateral triangles. How many edges and vertices does it have?

   Solution:

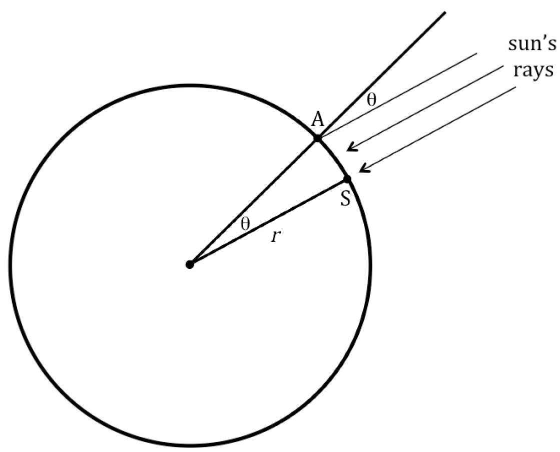
   A regular octahedron has 8 faces, 6 vertices, and 12 edges.



4. On the summer solstice, Eratosthenes observed that the sun's rays shown vertically and directly into a well at Syrene. In Alexandria, at the same date and time, the sun cast a shadow, and Eratosthenes calculated that the sun's rays fell at an angle of 7.2° from vertical. If Alexandria and Syrene are 490 miles apart, use this information to calculate the radius of the Earth.



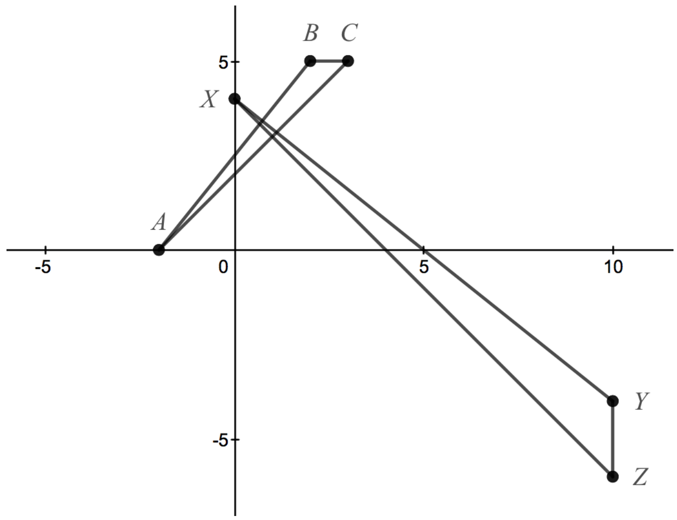
Solution: We assume the rays from the sun are parallel. The angle of the sun's ray from vertical must be the same as the angle between Alexandria and Syrene.



If the arc (distance) between Alexandria and Syrene is 490 miles, we can obtain the radius r using the formula for arc length $S = r\theta$ where θ is given in radians. Thus, $7.2^\circ = 7.2^\circ \left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{25}$, and thus

$$r = \frac{S}{\theta} = \frac{490}{\frac{\pi}{25}} = \frac{12250}{\pi}.$$

5. Let $\triangle ABC$ have vertices at $A = (-2, 0)$, $B = (2, 5)$, and $C = (3, 5)$, and let $\triangle XYZ$ have vertices at $X = (0, 4)$, and $Y = (10, -4)$, and $Z = (10, -6)$. Are these triangles similar? If so, what is the scaling factor?



Solution: Using the distance formula, yields

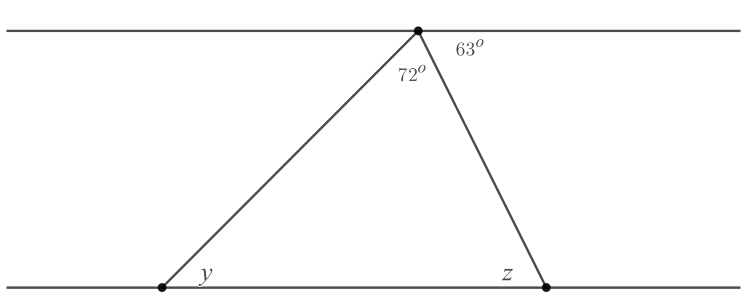
$$\begin{aligned}
AB &= \sqrt{(-2-2)^2 + (0-5)^2} = \sqrt{41} \\
BC &= 1 \\
CA &= \sqrt{(-2-3)^2 + (0-5)^2} = \sqrt{50} = 5\sqrt{2} \\
XY &= \sqrt{(0-10)^2 + (4-(-4))^2} = \sqrt{164} = 2\sqrt{41} \\
YZ &= 2 \\
ZX &= \sqrt{(10-0)^2 + (-6-4)^2} = \sqrt{200} = 10\sqrt{2}
\end{aligned}$$

To determine if the triangles are similar, we find the ratio between corresponding pairs of edges:

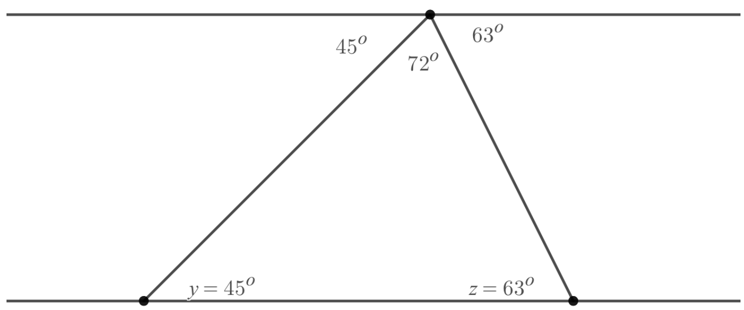
$$\frac{XY}{AB} = \frac{2\sqrt{41}}{\sqrt{41}} = 2 \qquad \frac{YZ}{BC} = \frac{2}{1} = 2 \qquad \frac{ZX}{CA} = \frac{10\sqrt{2}}{5\sqrt{2}} = 2$$

Because all of these ratios are equal, the triangles are similar with a scaling factor of 2.

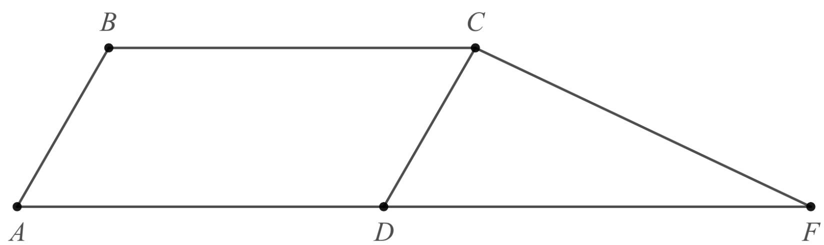
6. The vertices of a triangle lie on two parallel lines as shown in the figure below. Given the values of 63° and 72°, determine the angles y and z.



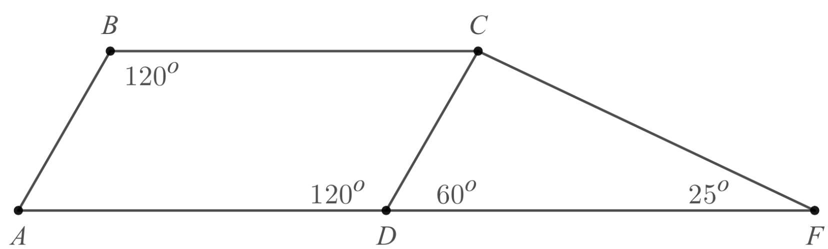
Solution: The top parallel line is divided into three angles, which must add to 180°. Then, the remaining angle at the top is $180° - 72° - 63° = 45°$. We can then use alternate interior angle theorem to find y and z:



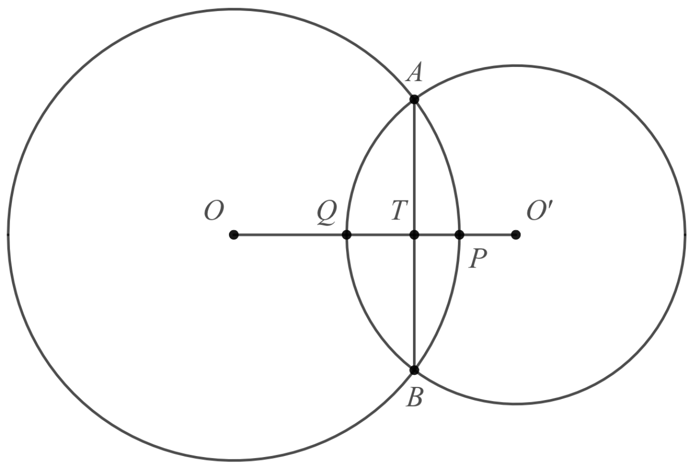
7. Assume that $ABCD$ in the following diagram is a parallelogram and that the points A, D, and F are collinear. If $\angle B = 120°$ and $\angle F = 25°$, find $\angle DCF$.



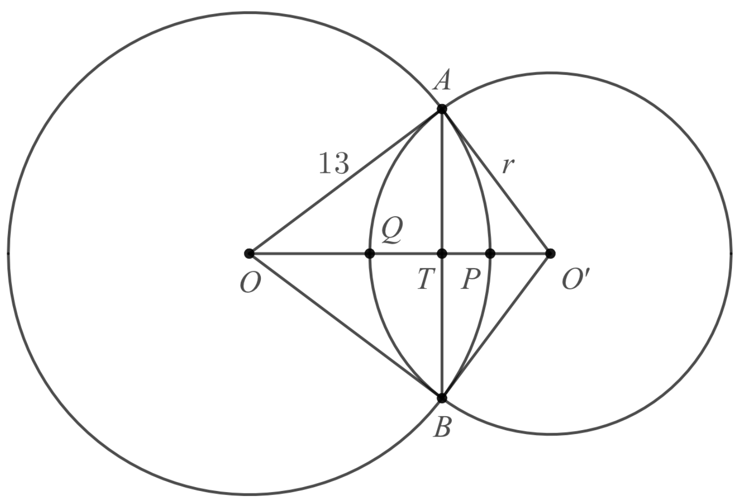
Solution: Because $ABCD$ is a parallelogram, opposite angles must be similar. Then, we can find $\angle CDF = 60°$, and $\angle DCF = 180° - 60° - 25° = 95°$.



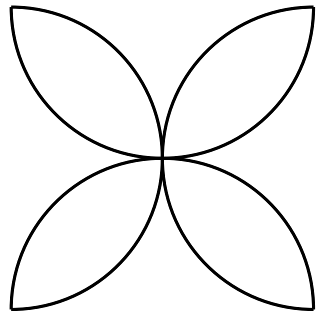
8. Two circles with centers of O and O' intersect at A and B. The length of the common chord $\overline{AB} = 10$. The line joining the centers intersects the circles at P and Q. If $\overline{PQ} = 2$ and the radius of one circle is 13, find the radius of the other circle. (Figure not drawn to scale.)



Solution: We let the circle centered at O have the radius of 13. Then, $\overline{OA} = \overline{OB} = 13$. The line OO' is a perpendicular bisector of AB, making $\overline{AT} = 5$. Using the Pythagorean theorem on $\triangle ATO$, which has hypotenuse 13 and a side of 5, we find $\overline{OT} = 12$. Because OP is also a radius of the left circle, $\overline{OP} = 13$, and $\overline{PT} = \overline{OP} - \overline{OT} = 1$. If $\overline{PQ} = 2$, then $\overline{QT}$ is also 1. Through symmetry, the circles must have the same radius of 13.

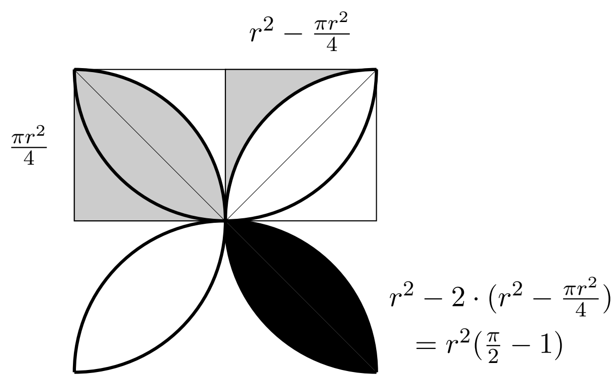


9. The intersction of four semicircles of radius r defines a flower shape with four petals. What is the radius r if the total area of the flower is 2.



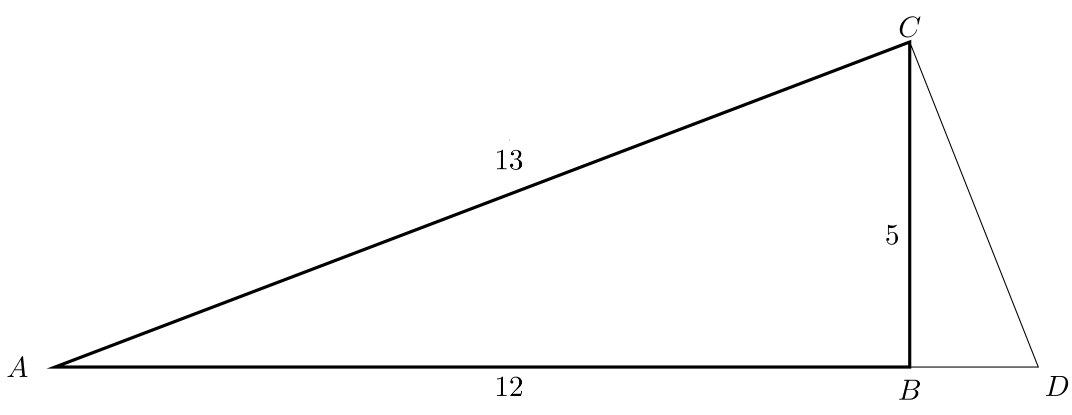
Solution:

The area of one petal is computed in three steps, illustrated clockwise from the upper left.

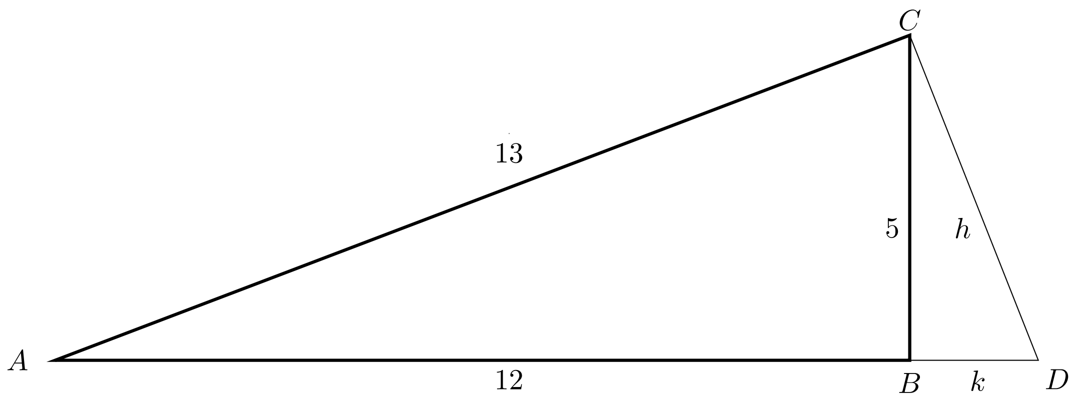


The area of the flower is then $4r^2(\frac{\pi}{2} - 1) = 2$, and therefore $r = \frac{1}{\sqrt{\pi - 2}}$.

10. A triangle ABC with side lengths of 12, 5, and 13 is extended to a similar triangle ACD as shown below. What is the area of the larger triangle ACD?



Solution: We begin by computing the lengths of the sides of the triangle in the extension.



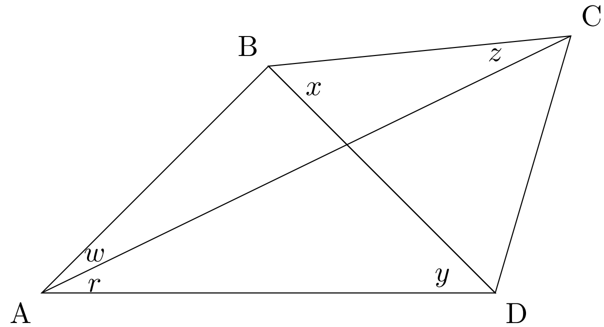
Since ABC and ACD are similar, BCD is also similar and so the corresponding sides of the triangles ABC and CBD are proportional. We solve the equations:

$\frac{12}{5} = \frac{13}{h} = \frac{5}{k}$,

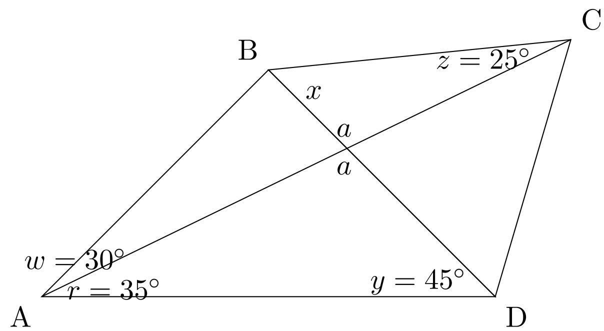
for k and h to get $k = 25/12$ and $h = 65/12$.

Second, we compute the area of triangle $ADC = \frac{1}{2}(12 + k) \cdot 5 = \frac{1}{2}(12 + \frac{25}{12})5 = 30 + \frac{125}{24} = \frac{845}{24}$

11. Trapezoid ABCD has the diagonals connecting the corners and the resulting angles are: $w = 30°$, $r = 35°$, $y = 45°$, and $z = 25°$, where positions of r, w, x, y and z are shown in the figure below. Find the value of the angle labeled x.

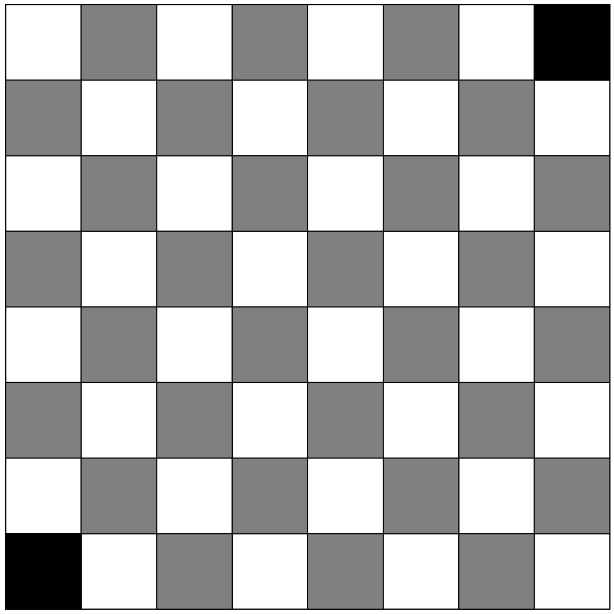


Solution: Given $w = 30°$, $r = 35°$, $y = 45°$, and $z = 25°$ we have the figure below.

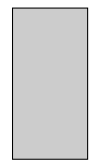


There is unnecessary information provided in this problem, so there are many ways to solve it. What follows is one possible solution. We know that $r + y + a = 180°$, so $a = 100°$. A similar argument shows that $180 - a - z = x$, and $x = 55°$.

12. How many 2×1 tiles can be placed on an 8×8 "checker board" after two of the opposite corners of the board have been removed?

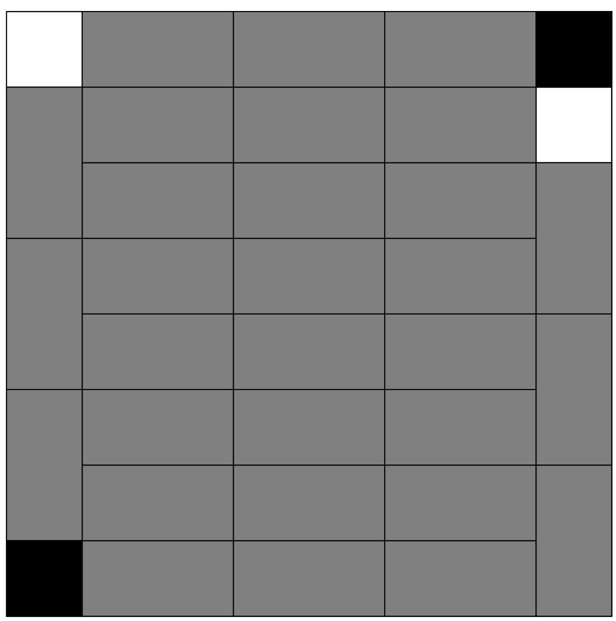


Board



Tile

Solution: Each tile will cover one white square and one gray square, since two gray squares have been removed there are 30 gray squares left on the board. You can fit 30 tiles on the board. There will be two white squares left uncovered.



2022 West River Math Contest
Geometry – Exam II

Record the correct answer on the answer sheet.

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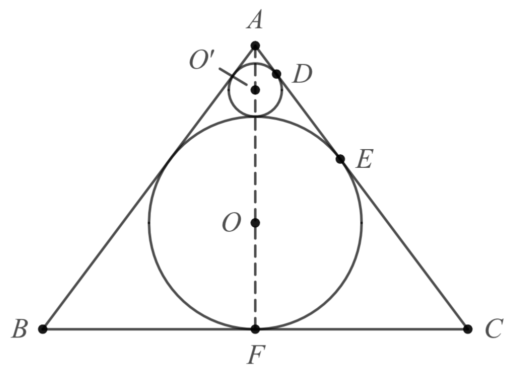
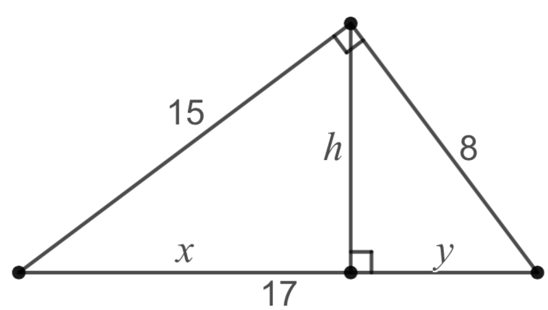
1. A circle is inscribed in a triangle with sides of lengths 10, 10, and 12. A second, smaller circle is inscribed tangent to the first circle and tangent to the equal sides of the triangle. Find the measure of the radius of the second, smaller circle.

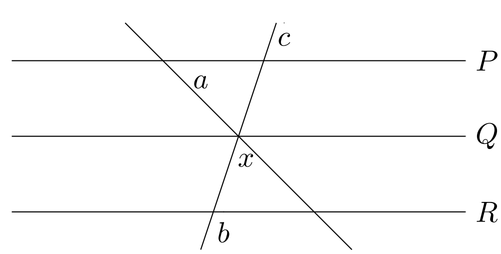


2. Given an 8–15–17 right triangle, as shown, determine the height of the perpendicular bisector h. (Figure not drawn to scale.)

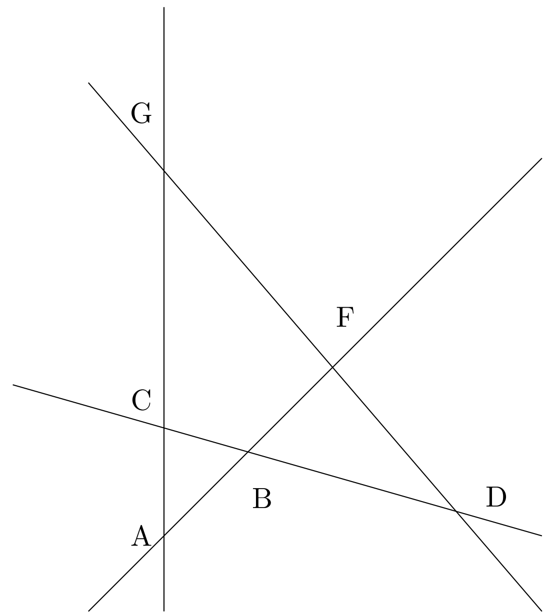


3. If we place as many non-overlapping unit disks in a 20×20 square as possible, what would be the area of the remaining un-covered portion of the square?

4. Given three parallel lines P, Q and R and the 2 non-parallel lines as shown below. Find the angle x if the angles a, b, c are related as follows:
$b = 2a + 12$ and $3 + \frac{b}{2} = c$

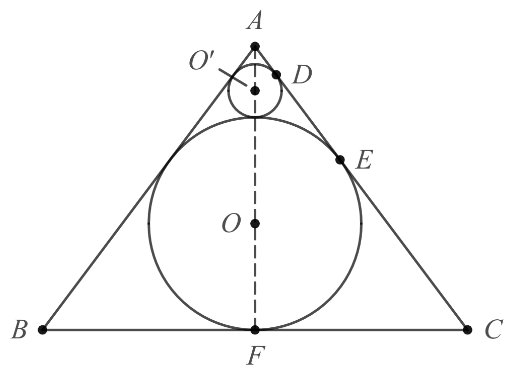


5. How many can triangles can be drawn by drawing 7 lines? For example 4 lines can create 4 triangles, the triangles below are (ABC), (BFD), (AFG), and (CDG).

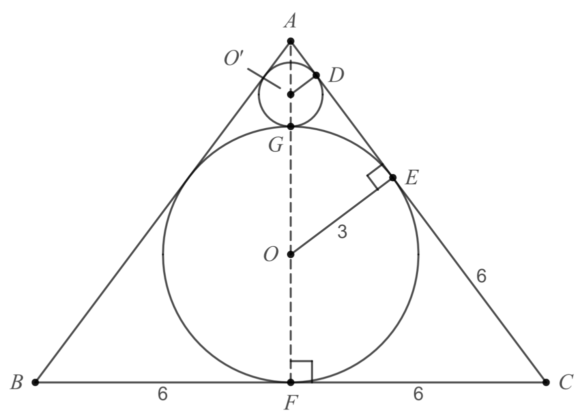


2022 West River Math Contest
Geometry – Exam II
Solutions

1. A circle is inscribed in a triangle with sides of lengths 10, 10, and 12. A second, smaller circle is inscribed tangent to the first circle and tangent to the equal sides of the triangle. Find the measure of the radius of the second, smaller circle.



Solution: Because line AC is tangent to the circles centered at O and O', lines OE and $O'D$ are perpendicular to AC. The dotted line AF is a perpendicular bisector of the isosceles triangle, so $\overline{CF} = 6$. Tangent segments drawn to a circle from the same point C must be equal; therefore, $\overline{CF} = \overline{CE} = 6$. Since $\overline{AC} = 10$, $\overline{AE} = 4$.



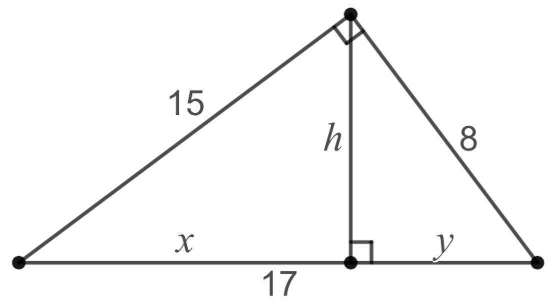
We now consider similar right triangles $\triangle AEO$ and $\triangle AFC$:

$$\frac{CF}{OE} = \frac{AF}{AE} \quad \Rightarrow \quad \frac{6}{OE} = \frac{8}{4}$$

and $\overline{OE} = 3$. The height AF of right $\triangle AFC$ is 8. Then, from the circle radius of $\overline{OE} = 3$, we find the diameter $\overline{GF} = 6$ and the remaining distance $\overline{AG} = 2$. To find the radius of the small circle, we let $O'D = O'G = r$ and then $O'A = 2 - r$. We know $AO = AG + GO$, so $\overline{AO} = 5$. Now, using similar triangles $\triangle ADO'$ and $\triangle AEO$:

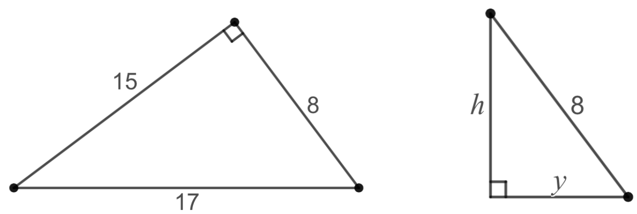
$$\frac{O'D}{O'A} = \frac{OE}{AO} \quad \Rightarrow \quad \frac{r}{2-r} = \frac{3}{5} \quad \Rightarrow \quad 5r = 6 - 3r \quad \Rightarrow \quad r = \frac{3}{4}$$

2. Given an 8–15–17 right triangle, as shown, determine the height of the perpendicular bisector h. (Figure not drawn to scale.)



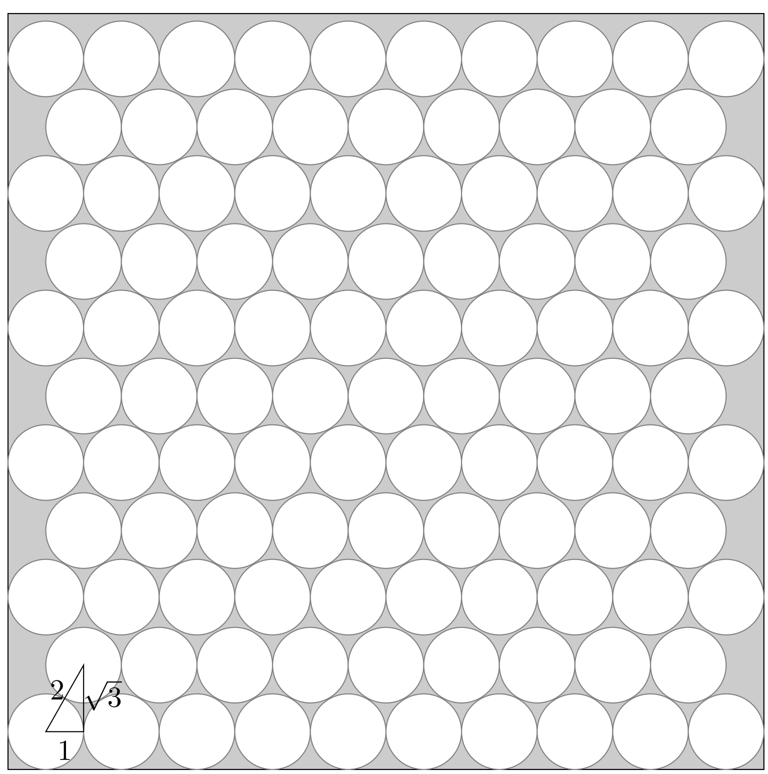
Solution: We can use similar triangles to find h. Specifically,

$$\frac{h}{8} = \frac{15}{17} \qquad \Rightarrow \qquad h = \frac{(15)(8)}{17} = \frac{120}{17}$$



3. If we place as many non-overlapping unit disks in a 20×20 square as possible, what would be the area of the remaining un-covered portion of the square?

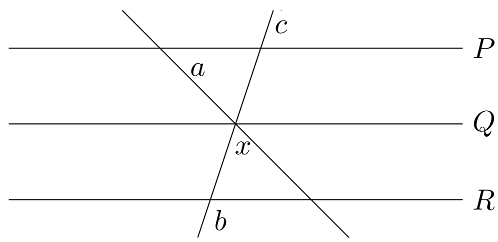
Solution:



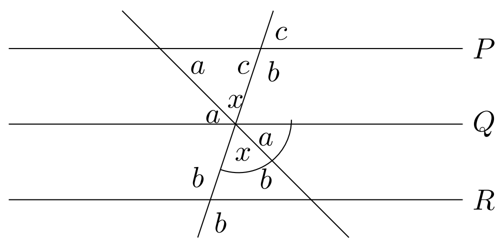
If the number of horizontal rows of stacked circles is n, then the height of the circles is $2 + (n-1) \cdot \sqrt{3}$. Therefore, $(n-1) \cdot \sqrt{3} \leq 18$, and since $\sqrt{3} < 1.8$, we know that $10 \cdot \sqrt{3} < 10 \cdot 1.8 = 18$. This means that 11 rows of non-overlapping circles will fit in the square. Of these, 5 rows will contain 9 circles, while 6 will contain 10 circles and thus a total of $45 + 60 = 105$ circles fit into the square. The left over area is therefore $A = 20^2 - 105 \cdot \pi = 400 - 105\pi$.

4. Given three parallel lines P, Q and R and the 2 non-parallel lines as shown below. Find the angle x if the angles a, b, c are related as follows:

   $b = 2a + 12$ and $3 + \frac{b}{2} = c$



Solution: The alternate interior angle and the vertical angle theorems give the relations shown below.



We have the following 4 equations in 4 unknowns:

1. $b = 2a + 12$
2. $3 + \dfrac{b}{2} = c$
3. $b = x + a$
4. $x + a + c = 180^\circ$

Solving these for a, b, c and x we get:

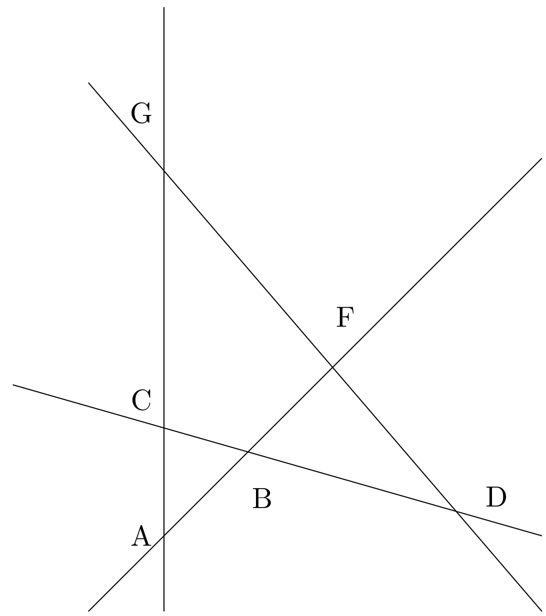
$a = 53^\circ$

$b = 118^\circ$

$c = 62^\circ$

$x = 65^\circ$

5. How many can triangles can be drawn by drawing 7 lines? For example 4 lines can create 4 triangles, the triangles below are (ABC), (BFD), (AFG), and (CDG).



Solution: There are at most 35 triangles created with 7 lines. If all of the lines are non-parallel and there are no three lines which intersect at one point then all of the 35 triangles will be produced. Every collection of three non-parallel lines form a triangle. If there are 7 mutually non- parallel lines with distinct intersections, then there are 7 ways to choose the first line, 6 ways to choose the second line and 5 ways to choose the third line. However, each triangle is then counted 6 times since there are six ways to order the 3 chosen lines. We can also describe this in the combinatorial fashion of 7 choose 3 which is $\binom{7}{3} = \frac{7!}{(7-3)!3!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ triangles.

2022 West River Math Contest
Advanced – Exam I

Record the correct answer on the answer sheet.

---

1. Which of the numbers 2^{100}, 3^{75}, 5^{50} is largest?

2. A man has \$2.73 in pennies, nickels, dimes, quarters, and half dollars. What is the total number of coins he has, if he has an equal number of coins of each kind?

3. If $ax + 3y = 5$ and $2x + by = 3$ represent the same line, determine the values of a and b.

4. The expression $\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}$ may be written as $\sqrt{n}$ for n a positive integer. What is n?

5. For the linear function $f(x) = ax + b$ determine all values of a and b so that $f(x) - f^{-1}(x) = 2020$. *Note:* The symbol f^{-1} denotes the inverse function of f, not its reciprocal.

6. Determine the smallest positive solution, in exact radians, to the following equation:
$$5\cos(x) + 2\,[\sin(x)]^2 = 4.$$

7. Write 2020 as a base 5 number.

8. Write the repeating decimal $0.\overline{123}$ as a fraction.

9. Simplify $(\log_2 3)(\log_9 16)$. Your answer should not involve any logarithms.

10. Suppose we multiply two integers.
    (a) If one integer has two digits and the other has three digits, what is the smallest number of digits possible in their product? (Suppose the leading digits of the integers are non-zero throughout this problem.)
    (b) If one integer has two digits and the other has three digits, what is the largest number of digits possible in their product?
    (c) If one integer has $m \geq 1$ digits and the other has $n \geq 1$ digits, what is the smallest number of digits possible in their product?
    (d) If one integer has $m \geq 1$ digits and the other has $n \geq 1$ digits, what is the largest number of digits possible in their product?

11. Find all solutions for x in the following absolute value equation:
$$|2x+7| + |3x-5| = 11$$

12. According to *Zipf's Law*, for any work by any author, it is approximately true that word rank multiplied by word frequency is equal to a constant. In Dickens' *Tale of Two Cities*, the word 'a' is fifth most common (5 is its rank) and occurs 2,944 times (2,944 is its frequency). It turns out that the word 'that' is tenth most common. Using Zipf's Law, estimate the frequency of 'that' in *Tale of Two Cities.*

13. *Benford's Law*, when it holds for a population of numbers, states the leading nonzero digit D, necessarily from 1 through 9, will be d with chance

$$P(D = d) = \log\left(\frac{d+1}{d}\right),$$

where *log* refers to the base 10 logarithm. This Law has, for example, been used in financial settings for fraud detection (e.g. in tax returns). Supposing this Law holds:

(a) Which leading nonzero digit (from 1 through 9) is most likely?

(b) Give a simple numeric answer for $P(1) + P(2) + \cdots + P(9)$ not involving logarithms.

(c) Provide a simplified formula for $P(D \le n)$, where n is understood to be a number from 1 through 9. (No credit for expressing your answer as a sum, for instance.)

2022 West River Math Contest
Advanced – Exam I
Solutions

---

1. Which of the numbers 2^{100}, 3^{75}, 5^{50} is largest?

   Solution: $2^{100} = 2^{4\cdot 25} = 16^{25}$, $3^{75} = 3^{3\cdot 25} = 27^{25}$ and $5^{50} = 5^{2\cdot 25} = 25^{25}$ so 3^{75} is largest.

2. A man has \$2.73 in pennies, nickels, dimes, quarters, and half dollars. What is the total number of coins he has, if he has an equal number of coins of each kind?

   Solution: If x is the number of coins of each kind, then $x + 5x + 10x + 25x + 50x = 273$. So, $x = 3$ and we have 15 coins in total.

3. If $ax + 3y = 5$ and $2x + by = 3$ represent the same line, determine the values of a and b.

   Solution: The two equations given may be rewritten as $3ax + 9y = 15$ and $10x + 5by = 15$. Equating coefficients of x and y in turn gives $a = 10/3$ and $b = 9/5$.

4. The expression $\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}$ may be written as $\sqrt{n}$ for n a positive integer. What is n?

   Solution: Squaring both sides of $\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}} = \sqrt{n}$ we obtain

   $$\begin{aligned} n &= \left(2+\sqrt{3}\right) + \left(2-\sqrt{3}\right) + 2\sqrt{2+\sqrt{3}}\sqrt{2-\sqrt{3}} \\ &= 4 + 2\sqrt{(2+\sqrt{3})(2-\sqrt{3})} \\ &= 4 + 2\sqrt{4-3}, \end{aligned}$$

   so $n = 6$.

5. For the linear function $f(x) = ax + b$ determine all values of a and b so that $f(x) - f^{-1}(x) = 2020$. *Note:* The symbol f^{-1} denotes the inverse function of f, not its reciprocal.

   Solution: Solving the relation $x = ay + b$ for y we find the inverse to be

   $$f^{-1}(x) = \frac{1}{a}x - \frac{b}{a}$$

   (f does not have an inverse if $a = 0$). This gives

   $$2020 = f(x) - f^{-1}(x) = \left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right).$$

   Since this must hold for all x, $a - 1/a = 0$ which implies either $a = 1$ or $a = -1$. Only $a = 1$ allows us to solve $b + b/a = 2020$, which then gives $b = 1010$.

6. Determine the smallest positive solution, in exact radians, to the following equation:

   $$5\cos(x) + 2\,[\sin(x)]^2 = 4.$$

   Solution: Replacing $[\sin(x)]^2$ by $1 - [\cos(x)]^2$ we find

   $$2\,[\cos(x)]^2 - 5\cos(x) + 2 = 0.$$

   or $0 = 2y^2 - 5y + 2 = (2y-1)(y-2)$ where $y = \cos(x)$. $y = \cos(x) = 2$ has no solution. The smallest positive solution to $y = \cos(x) = 1/2$ is $x = \pi/3$.

7. Write 2020 as a base 5 number.

Solution: Note that $5^0 = 1$, $5^1 = 5$, $5^2 = 25$, $5^3 = 125$, and $5^4 = 625$. Note 625 divides into 2020 *three* times with remainder 145. Now, 125 divides into 145 *one* time with remainder 20. Continuing, 25 divides into 20 *no* times with remainder 20, 5 divides into 20 *four* times with remainder zero. More succinctly, $2020 = 3 \cdot 625 + 1 \cdot 125 + 0 \cdot 25 + 4 \cdot 5 + 0 \cdot 1$. This gives the base 5 number 31040_5.

8. Write the repeating decimal $0.\overline{123}$ as a fraction.

Solution: Letting $x = 0.\overline{123}$, $1000x = 123.\overline{123}$. Subtracting, we find $999x = 123$ and $x = 123/999$ or $41/333$.

9. Simplify $(\log_2 3)(\log_9 16)$. Your answer should not involve any logarithms.

Solution: One approach to this is using the change of base formula for logarithms:

$$(\log_2 3)(\log_9 16) = \frac{\ln 3}{\ln 2} \cdot \frac{\ln 16}{\ln 9} = \frac{\ln 3}{\ln 2} \cdot \frac{\ln 2^4}{\ln 3^2} = 2.$$

10. Suppose we multiply two integers.

(a) If one integer has two digits and the other has three digits, what is the smallest number of digits possible in their product? (Suppose the leading digits of the integers are non-zero throughout this problem.)

(b) If one integer has two digits and the other has three digits, what is the largest number of digits possible in their product?

(c) If one integer has $m \geq 1$ digits and the other has $n \geq 1$ digits, what is the smallest number of digits possible in their product?

(d) If one integer has $m \geq 1$ digits and the other has $n \geq 1$ digits, what is the largest number of digits possible in their product?

Solution:

(a) Since the smallest such product, $10 \times 100 = 1,000$, four is the smallest number of digits.

(b) Since the largest such product, $99 \times 999 = 98,901$, five is the largest number of digits.

(c) The product can be as small as $10^{m-1} \cdot 10^{n-1} = 10^{m+n-2}$, which is a 1 followed by $m + n - 2$ zeroes. Consequently, $m + n - 1$ is the smallest number of digits.

(d) The largest product is strictly less than $10^m \cdot 10^n = 10^{m+n}$, which is a one followed by $m + n$ zeroes. Consequently, the product has no more than $m + n$ digits. That the largest number of digits *is* $m + n$ can be seen, for example, by choosing each number to consist of a 9 followed by zeroes.

Note: The answers to parts (c) and (d) might reasonably be guessed from the answers in parts (a) and (b).

11. Find all solutions for x in the following absolute value equation:

$$|2x + 7| + |3x - 5| = 11$$

Solution: Note that the linear terms inside absolute values, $2x + 7$ and $3x - 5$, are zero at -7/2 and 5/3, respectively. It suffices to look at three cases:

Case 1: $x \leq -7/2$
$11 = -(2x+7) + -(3x-5) = -5x - 2$, giving $x = -13/5$. We exclude this 'solution' as it doesn't satisfy $x \leq -7/2$.

Case 2: $-7/2 < x < 5/3$
$11 = (2x+7) + -(3x-5) = -x + 12$, giving $x = 1$.

Case 3: $x \geq 5/3$
$11 = (2x+7) + (3x-5) = 5x + 2$, giving $x = 9/5$.

Consequently, $x = 1$ and $x = 9/5$ are the solutions.

12. According to *Zipf's Law*, for any work by any author, it is approximately true that word rank multiplied by word frequency is equal to a constant. In Dickens' *Tale of Two Cities*, the word 'a' is fifth most common (5 is its rank) and occurs 2,944 times (2,944 is its frequency). It turns out that the word 'that' is tenth most common. Using Zipf's Law, estimate the frequency of 'that' in *Tale of Two Cities*.

    Solution: For the word 'a' we have $\mathit{Rank} \cdot \mathit{Frequency} = 5 \cdot 2944 = 14720$. Solving, for the word 'that,' $10(\mathit{Frequency}) = 14720$, we estimate that 'that' appears 1,472 times in *Tale of Two Cities* ('that' actually appears 1,941 times).

13. *Benford's Law*, when it holds for a population of numbers, states the leading nonzero digit D, necessarily from 1 through 9, will be d with chance

$$P(D = d) = \log\left(\frac{d+1}{d}\right),$$

    where *log* refers to the base 10 logarithm. This Law has, for example, been used in financial settings for fraud detection (e.g. in tax returns). Supposing this Law holds:

    (a) Which leading nonzero digit (from 1 through 9) is most likely?
    (b) Give a simple numeric answer for $P(1) + P(2) + \cdots + P(9)$ not involving logarithms.
    (c) Provide a simplified formula for $P(D \leq n)$, where n is understood to be a number from 1 through 9. (No credit for expressing your answer as a sum, for instance.)

    Solution:

    (a) $P(1) = \log(2)$ is largest. So '1' is the most common leading significant digit.
    (b) This is the chance that *some* digit appears. Consequently, this sum is 1. (This answer can also be obtained by working part (c) first and evaluating at $n = 9$.)
    (c) $P(D \leq n) = P(D = 1) + P(D = 2) + \cdots + P(D = n-1) + P(D = n)$ but using $\log((d+1)/d) = \log(d+1) - \log(d)$ this sum becomes $(\log(2) - \log(1)) + (\log(3) - \log(2)) + \cdots + (\log(n) - \log(n-1)) + (\log(n+1) - \log(n)) = \log(n+1) - \log(1) = \log(n+1)$.

Acknowledgement: Some of the problems are from the 2019 Lehigh University High School Math Contest (`https://www.lehigh.edu/~dmd1/scan19.pdf`) and from past Texas A & M University High School Mathematics contests
(`www.math.tamu.edu/outreach/highschoolcontest/`).

2022 West River Math Contest
Advanced – Exam II

Record the correct answer on the answer sheet.

---

1. A bus holds up to 45 people. It doesn't have to be full. The fare for adults is \$8, and the fare for children is \$5. If the total paid by all riders is exactly \$250, what is the maximum possible number of children on the bus?

2. Jane can mow a field in 12 hours. Jane and Bill together can mow this field in 8 hours. How long would it take Bill (alone) to mow this field?

3. If
$$\frac{x+y}{x-y} + \frac{x-y}{x+y} = 3,$$
then determine the value of
$$\frac{x^2+y^2}{x^2-y^2} + \frac{x^2-y^2}{x^2+y^2}.$$

4. Fibonacci numbers are given by the recursion relation
$$F(n+2) = F(n) + F(n+1),$$
with starting values $F(1) = 1$, $F(2) = 1$. So, $F(3) = F(1) + F(2) = 2$ and $F(4) = F(2) + F(3) = 3$, for example. What happens to the ratio
$$\frac{F(n+1)}{F(n)}$$
as n becomes arbitrarily large?

5. Express, as a simple fraction, the value of
$$\frac{1}{1(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{2019(2020)} + \frac{1}{2020(2021)}.$$
Hint: For some values of constants A and B,
$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{(n+1)}.$$
Start by finding A and B.

2022 West River Math Contest
Advanced – Exam II
Solutions

---

1. A bus holds up to 45 people. It doesn't have to be full. The fare for adults is \$8, and the fare for children is \$5. If the total paid by all riders is exactly \$250, what is the maximum possible number of children on the bus?

   Solution: If we let A denote the number of adults and C denote the number of children, then $8A + 5C = 250$ or

$$A = \frac{5}{8}(50 - C).$$

   This implies $50 - C$ must be a multiple of 8. The table below shows the possibilities with the largest values of C ignoring bus capacity.

$50 - C$	C	$A = 5(50 - C)/8$	$A + C$
0	50	0	50
8	42	5	47
16	34	10	44
24	26	15	41
⋮	⋮	⋮	⋮

   Since the maximum capacity of the bus is 45 passengers—eliminating the first two rows of the table, we find 34 to be the maximal number of children (with 10 adults) on the bus.

2. Jane can mow a field in 12 hours. Jane and Bill together can mow this field in 8 hours. How long would it take Bill (alone) to mow this field?

   Solution: In 8 hours, Jane can mow $8/12 = 2/3$ of the field, and Bill can mow $1/3$ of the field. Therefore, it would take Bill 24 hours to mow the field alone.

3. If

$$\frac{x+y}{x-y} + \frac{x-y}{x+y} = 3,$$

   then determine the value of

$$\frac{x^2+y^2}{x^2-y^2} + \frac{x^2-y^2}{x^2+y^2}.$$

   Solution: Getting a common denominator

$$3 = \frac{x+y}{x-y} + \frac{x-y}{x+y} = \frac{(x+y)^2}{x^2-y^2} + \frac{(x-y)^2}{x^2-y^2} = \frac{2(x^2+y^2)}{x^2-y^2},$$

   so $\frac{(x^2+y^2)}{x^2-y^2} = \frac{3}{2}$, $\frac{x^2-y^2}{(x^2+y^2)} = \frac{2}{3}$, and the desired sum is $\frac{3}{2} + \frac{2}{3}$ or $\frac{13}{6}$.

4. Fibonacci numbers are given by the recursion relation

$$F(n+2) = F(n) + F(n+1),$$

with starting values $F(1) = 1$, $F(2) = 1$. So, $F(3) = F(1) + F(2) = 2$ and $F(4) = F(2) + F(3) = 3$, for example. What happens to the ratio

$$\frac{F(n+1)}{F(n)}$$

as n becomes arbitrarily large?

Solution: Dividing both sides of

$$F(n+2) = F(n) + F(n+1)$$

by $F(n+1)$, we get

$$\frac{F(n+2)}{F(n+1)} = \frac{F(n)}{F(n+1)} + 1.$$

Letting r be the value of $\frac{F(n+1)}{F(n)}$ as n gets arbitrarily large, we have

$$r = \frac{1}{r} + 1.$$

Solving this we get

$$r = \frac{1 \pm \sqrt{5}}{2}.$$

Because the limiting ratio should be positive, we take the positive root in the above.

5. Express, as a simple fraction, the value of

$$\frac{1}{1(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{2019(2020)} + \frac{1}{2020(2021)}.$$

Hint: For some values of constants A and B,

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{(n+1)}.$$

Start by finding A and B.

Solution: Multiplying both sides of

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{(n+1)}$$

by $n(n+1)$, we have

$$1 = (n+1)A + nB = n(A+B) + A,$$

so $A = 1$ and $A + B = 0$, giving $A = 1$ and $B = -1$. Consequently,

$$\frac{1}{1(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{2019(2020)} + \frac{1}{2020(2021)}$$

becomes

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{2019}-\frac{1}{2020}\right)+\left(\frac{1}{2020}-\frac{1}{2021}\right).$$

All of the terms in this sum, with the exception of the first and last, cancel, leaving $1-\frac{1}{2021}$ or $\frac{2020}{2021}$.

Acknowledgement: Some of the problems are from the 2019 Lehigh University High School Math Contest (`https://www.lehigh.edu/~dmd1/scan19.pdf`) and from past Texas A & M University High School Mathematics contests
(`www.math.tamu.edu/outreach/highschoolcontest/`).

2022 West River Math Contest
Masters – Exam

Record the correct answer on the answer sheet.

---

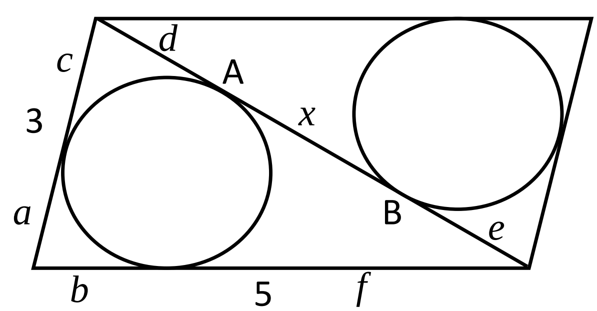
1. Spot's doghouse has a regular hexagonal base that measures one yard on each side. He is tied to a vertex with a rope that is two yards long. What is the area, in square yards, of the region outside the doghouse that Spot can reach? (You may assume that Spot is just a point with no length, width, or height.)

2. A permutation $(a_1, a_2, a_3, a_4, a_5)$ of the integers $1, 2, 3, 4, 5$ is *heavy-tailed* if $a_1 + a_2 < a_4 + a_5$. For example, the permutation $(2, 4, 1, 5, 3)$ is heavy-tailed, since $2 + 4 < 5 + 3$. How many heavy-tailed permutations of $1, 2, 3, 4, 5$ are there?

3. The Colorado Rockies play the Arizona Diamondbacks in a National League Division Series, which is a "best of five" game series. How many outcomes are possible in which the Rockies win the series? (Example: one possible outcome in which the Rockies win the series is RRDR, indicating that the Rockies win games 1, 2, and 4 and the Diamondbacks win game 3.)

4. Consider the equation $2^{2x} - 3^{2y} = 55$, where x and y are positive integers. List all pairs (x, y) that solve this equation.

5. A parallelogram with sides of length 3 and 5 units is divided into two triangles by its diagonal, as shown in the figure. A circle is inscribed in each triangle. Determine the distance x between points A and B, the points at which the circles are tangent to the diagonal. (The symbols a, b, etc. are the lengths of the various segments in the figure.)



6. Recall that n factorial is defined by

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1.$$

When the number 12! is expressed in base 12, the last several "digits" will be zeros. Exactly how many such zeros will there be?

2022 West River Math Contest
Masters – Exam
Solutions

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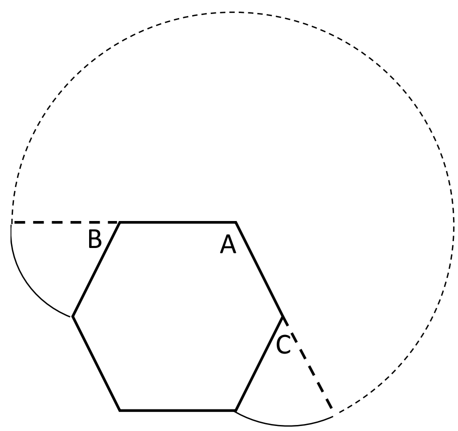
1. Spot's doghouse has a regular hexagonal base that measures one yard on each side. He is tied to a vertex with a rope that is two yards long. What is the area, in square yards, of the region outside the doghouse that Spot can reach? (You may assume that Spot is just a point with no length, width, or height.)

   Solution: In the figure below, the interior angle A of the regular hexagon measures $120°$, while the angles B and C each measure $60°$. If Spot's leash is anchored at the vertex corresponding to angle A, the circular arcs represent the boundaries of the region Spot can reach. The area of the region bounded by the dashed circular arc (with radius 2 yards) is $\pi \cdot 2^2 \cdot \frac{240°}{360°}$, or $\frac{8\pi}{3}$ square yards. Each region bounded by a small, solid circular arc (with radius 1 yard) is $\pi \cdot 1^2 \cdot \frac{60°}{360°}$, or $\frac{\pi}{6}$ square yards. The total area that spot can reach is therefore $\frac{8\pi}{3} + \frac{\pi}{6} + \frac{\pi}{6} = 3\pi$ square yards.



2. A permutation $(a_1, a_2, a_3, a_4, a_5)$ of the integers $1, 2, 3, 4, 5$ is *heavy-tailed* if $a_1 + a_2 < a_4 + a_5$. For example, the permutation $(2, 4, 1, 5, 3)$ is heavy-tailed, since $2 + 4 < 5 + 3$. How many heavy-tailed permutations of $1, 2, 3, 4, 5$ are there?

   Solution: We consider five cases for a_3.

   Suppose $a_3 = 1$. The sum of the remaining numbers is $a_1 + a_2 + a_4 + a_5 = 14$, so we will have a heavy-tailed permutation if $a_1 + a_2 < 7$ (which would make $a_4 + a_5 > 7$, so $a_1 + a_2 < a_4 + a_5$). Choosing 2 and 3 gives a sum less than 7; there are four heavy-tailed permutations for this choice: (a_1, a_2) could be $(2, 3)$ or $(3, 2)$, and each of these could have $(a_4, a_5) = (4, 5)$ or $(5, 4)$. Critical observation: *each pair with sum less than 7 produces four heavy-tailed permutations.* Thus four more heavy-tailed permutations result from the pair $(2, 4)$. (Total of 8.)

   If $a_3 = 2$ the sum of the remaining numbers is 13, so we need $a_1 + a_2 < 6.5$. The three pairs that work here are $(1, 3), (1, 4), (1, 5)$, each producing four heavy-tailed permutations. (Total of 12.)

   If $a_3 = 3$ the sum of the remaining numbers is 12, so we need $a_1 + a_2 < 6$. The two pairs that work here are $(1, 2)$ and $(1, 4)$, each producing four heavy-tailed permutations. (Total of 8.)

If $a_3 = 4$ the sum of the remaining numbers is 11, so we need $a_1 + a_2 < 5.5$. The three pairs that work here are $(1, 2), (1, 3)$ and $(2, 3)$, each producing four heavy-tailed permutations. (Total of 12.)

If $a_3 = 5$ the sum of the remaining numbers is 10, so we need $a_1 + a_2 < 5$. The two pairs that work here are $(1, 2)$ and $(1, 3)$, each producing four heavy-tailed permutations. (Total of 8.)

There are a grand total of 48 heavy-tailed permutations!

3. The Colorado Rockies play the Arizona Diamondbacks in a National League Division Series, which is a "best of five" game series. How many outcomes are possible in which the Rockies win the series? (Example: one possible outcome in which the Rockies win the series is RRDR, indicating that the Rockies win games 1, 2, and 4 and the Diamondbacks win game 3.)

   Solution: There is exactly one way for the Rockies to win in three games: They must win all three.

   For the Rockies to win in four games, the last game must be a Rockies win; additionally the Rockies must win 2 of the first 3 games. There are $\binom{3}{2} = 3$ ways for this to occur.

   For the Rockies to win in five games, the last game must be a Rockies win; additionally the Rockies must win 2 of the first 4 games. There are $\binom{4}{2} = 6$ ways for this to occur.

   Thus there are a total of $1 + 3 + 6 = 10$ outcomes in which the Rockies win the Division Series.

4. Consider the equation $2^{2x} - 3^{2y} = 55$, where x and y are positive integers. List all pairs (x, y) that solve this equation.

   Solution: $2^{2x} - 3^{2y}$ can be factored as a difference of two squares, so that

$$(2^x + 3^y)(2^x - 3^y) = 55.$$

   Since the only positive integer factors of 55 are 1, 5, 11, and 55, and since $2^x + 3^y$ is the larger factor, we must have

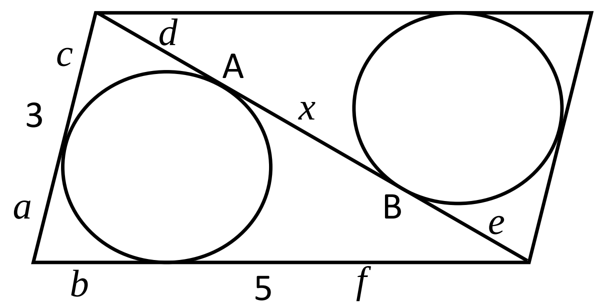
$$2^x + 3^y = 11$$
$$2^x - 3^y = 5$$

   or

$$2^x + 3^y = 55$$
$$2^x - 3^y = 1.$$

   The first of these systems solves easily by adding the two equations to produce $2 \cdot 2^x = 16$, or $2^x = 8$, which gives $x = 3$ and $y = 1$. Adding equations in the second system produces $2 \cdot 2^x = 56$, or $2^x = 28$. This has no integer solution. Thus the only solution is $x = 3$ and $y = 1$.

5. A parallelogram with sides of length 3 and 5 units is divided into two triangles by its diagonal, as shown in the figure. A circle is inscribed in each triangle. Determine the distance x between points A and B, the points at which the circles are tangent to the diagonal. (The symbols a, b, etc. are the lengths of the various segments in the figure.)



Solution: Consider the modification of the original figure as seen in Figure 1. With the figure labeled as shown, we see that $a + c = 3$ and $b + f = 5$. Also, since the triangles represented with dashed lines are congruent right triangles, we have $a = b$, and analogously one can see that $c = d = e$, and $x + e = f$. This gives us

$$x = f - e = 5 - b - e = 5 - a - c = 5 - (a + c) = 5 - 3 = 2.$$

6. Recall that n factorial is defined by

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1.$$

When the number 12! is expressed in base 12, the last several "digits" will be zeros. Exactly how many such zeros will there be?

Solution: If a base ten number is divisible by 10, its last digit must be zero; if it is divisible by 10^2, the last two digits will be zeros, etc. So it is in base 12: if the number is divisible by 12, the last "digit" will be zero, if it is divisible by 12^2, the last two digits will be 0, etc. So the problem is to determine how many factors of 12 there are in 12!. Write 12! in terms of its prime factors:

$$\begin{aligned} 12! &= 2^2 \cdot 3 \cdot 11 \cdot 2 \cdot 5 \cdot 3^2 \cdot 2^3 \cdot 7 \cdot 2 \cdot 3 \cdot 5 \cdot 2^2 \cdot 3 \cdot 2 \cdot 1 \\ &= 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11. \end{aligned}$$

Observe that $2^{10} \cdot 3^5 = 4^5 \cdot 3^5 = 12^5$. There are 5 zeros at the end of the base 12 representation of 12!.

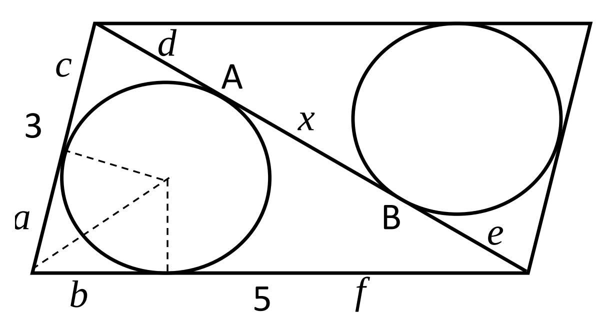


Figure 1: Modified figure for Problem 5.